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THE LENGTH OF HARMONIC FORMS ON A COMPACT RIEMANNIAN MANIFOLD

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ABSTRACT. We study $(n+1)$-dimensional Riemannian manifolds with harmonic forms of constant length and first Betti number equal to n showing that they are 2-step nilmanifolds with some special metrics. We also characterize, in terms of properties on the product of harmonic forms, the left-invariant metrics among them. This allows us to clarify the case of equality in the stable isosytolic inequalities in that setting. We also discuss other values of the Betti number.

1. Introduction

Let (M^n, g) be a compact oriented Riemannian manifold. Recently, the length of harmonic forms appeared to play a singular part in different geometric problems.

For example, in dimension 4, recent work of C. Lebrun [Leb02] shows a strong interplay between the length of harmonic self-dual 2-forms of the manifold and the non-vanishing of Seiberg-Witten invariants, in particular the existence of a symplectic structure.

Another setting where the length of harmonic forms naturally arises are the geometrically formal Riemannian manifolds. These are closed oriented Riemannian manifolds such that the product of two harmonic forms is still harmonic. They have the property that all harmonic forms have constant pointwise norm (see [Kot01]). In particular their first Betti number cannot be one less than the dimension and if it equals the dimension, then it is a flat torus.

The main goal of this note is to investigate, under some simple topological conditions, Riemannian metrics on compact oriented manifolds having the property that the length of any harmonic 1-form is (pointwise) constant. Our first result gives a precise metric and topological description of closed, oriented Riemannian manifolds all of whose harmonic 1-forms are of constant length, in the case when the first Betti number is one less than the dimension.

Theorem 1. *The compact oriented Riemannian manifolds (M^{n+1}, g) with $b_1(M) = n$ and all their harmonic 1-forms of constant length are in one-to-one correspondence*

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with triples (h, ω, c) *where:*

(1) h *is a flat metric on an* n*-torus* T^n*;*
(2) c *is a positive real constant;*
(3) ω *is a closed* 2*-form representing a non-zero integral class in the deRham cohomology group* $H^2_{DR}(T)$.

In fact, these metrics are explicitely given by a simple geometrical construction: given a triple (h, c, ω) as in Theorem 1, one uses Chern-Weil theory to get a principal $\mathbb{S}^1$-bundle $\mathbb{S}^1 \hookrightarrow N^{n+1} \xrightarrow{\pi} T$ equipped with a principal connexion form θ and whose curvature form is $\Omega = 2\pi\omega$. Theorem 1 then says that (M, g) is isometric with (N, g_θ), where the bundle-like metric g_θ is given by

$$g_\theta = \pi^\star h + c^2 \theta \otimes \theta.$$

A more detailed discussion of this family of metrics, which we call in what follows **minimal bundle-like** metrics, will be given in section 3. Note that, when M is fixed, the moduli space of metrics g satisfying the assumptions of the Theorem 1 is infinite dimensional, a fact significant in subsequent application. It is also worth mentioning that the manifolds N are in fact 2-step nilmanifolds with 1-dimensional center.

The relatively simple observation in Theorem 1 has a number of direct interesting implications, in at least two differently arising contexts, that we list below.

1.1. **The macroscopic spectrum of a nilmanifold** is given by the asymptotic behaviour of the eigenvalues of a Laplace-Beltrami operator acting on the function on the metric balls of the universal covering of a nilmanifold, as the radius of the balls goes to infinity. In [Ver02] the second author showed that the first eigenvalue of this macroscopic spectrum satisfies an inequality, whose equality case is attained by the nilmanifolds having all harmonic 1-forms of constant norm. This shows that the nilmanifolds with left-invariant metrics are not the only ones satisfying the equality case, as in the torus case. Hence Theorem 1 gives the following corollary in that setting:

Corollary 2. *Let* (M^{n+1}, g) *be a nilpotent Riemannian manifold, with first Betti number* $b_1 = n$*. Let* $B_g(\rho)$ *be the ball of radius* ρ *induced by the lifted metrics on the universal covering of* M^{n+1}*. Let* $\lambda_1\big(B_g(\rho)\big)$ *be the first eigenvalue of the laplacian acting on functions over* $B_g(\rho)$ *for the Dirichlet problem. Then there are some functions* $\lambda_1^\infty(g)$ *and* $\lambda_1^{al}(g)$ *such that*

$$\lim_{\rho\to\infty} \rho^2 \lambda_1\big(B_g(\rho)\big) = \lambda_1^\infty(g) \le \lambda_1^{al}(g)$$

with equality if and only if M^{n+1} *is a* 2*-step nilmanifold with 1-dimensional center and* g *a minimal bundle-like metric.*

1.2. Another application of Theorem 1 is related to the borderline case of **isosystolic stable inequalities**, studied recently by V. Bangert and M. Katz in [BK03a] (see also references therein), which give lower bounds on the volume of compact orientable manifolds in terms of some short closed geodesics (systols). More precisely, it is proven that for any closed oriented manifold (X^{n+1}, g) with $b_1(X) > 0$, the following holds:

$$\mathrm{stsys}_1(g)\mathrm{sys}_n(g) \le \gamma'_{b_1(X)} \mathrm{vol}_g(X), \tag{1}$$

where $\mathrm{stsys}_1(g)$ is the stable 1-systol of the metric g, $\mathrm{sys}_n(g)$ denotes the infimum of the n-volumes of all non-separating hypersurfaces in X^{n+1} and $\gamma'_{b_1(X)}$ is the Bergé-Martinet constant. If equality is attained, then all harmonic one-forms are of constant length [BK03b]. Hence Theorem 1 together with the precise description (see [BK03b]) of the Albanese torus of such a metric yields

Corollary 3. *Let (X^{n+1}, g) be a compact, oriented, $(n+1)$-dimensional Riemannian manifold with first Betti number $b_1(X) = n$. Then equality occurs in* (1) *if and only if X^{n+1} is a 2-step nilmanifold and g is a minimal bundle-like metric with base torus of the form $\mathbb{R}^n/L$, where L is a dual critical lattice in $\mathbb{R}^n$.*

Hence in the three-dimensional case, equality is only satisfied by the three-dimensional Heisenberg group endowed with a metric such that there is a Riemannian submersion onto an equilateral torus.

We are also able to characterize, in terms of product of harmonic forms (lead by the geometrically formal background), the left-invariant metrics among the minimal bundle-like metrics, in the case of the Betti number being one less than the dimension (see Theorems 10 to 12).

These last results might be compared with the work in [ACGR], where it is shown that an n-dimensional oriented manifold M having $n-1$ small (compared to the diameter) eigenvalues (for the laplacian acting on one-forms) is diffeomorphic to a nilmanifold with a metric close to a left-invariant one. In their paper the authors needed strong assumptions on the curvature. In this note, instead of a control of the curvature, we have a control of the length of harmonic forms. We would like to stress that this seems to be a hidden assumption on the curvature.

2. Implication of the existence of harmonic forms of constant length

2.1. **The Albanese map.** Let (M^n, g) be a compact oriented Riemannian manifold having all of its harmonic one forms of constant length.

We stress that we do not consider the case where there are harmonic forms of constant length, but the case when *all* harmonic forms are supposed to be of constant length. This implies for example that the pointwise scalar product of two harmonic forms is constant, which is not the case with just the existence assumption as the following example shows (which happens to answers question 7 of section 10 in [BK03b]):

Example 4. Let (M^n, g) be Riemannian manifold, T^2 a torus, and consider $N^{n+2} = M^n \times T^2$ endowed with the Riemannian metric $h = g \oplus s$, where s is defined as follows. If $f : M \to \mathbb{R}$ is a smooth function such that $f^2 < 1$, we define s on T^2 by setting $s(dx, dx) = 1 = s(dy, dy)$ and $s(dx, dy) = f$. We claim that (the closed) 1-forms $\alpha_1 = dx$ and $\alpha_2 = dy$ are also co-closed thus harmonic, but by construction, their scalar product is not constant.

Proof. Let us take $(e_i)_{1\leq i\leq n+2}$ a local orthonormal basis of N^{n+2}, with $(e_k)_{1\leq k\leq n}$ spanning TM, and e_{n+1}, e_{n+2} spanning TT^2. Consider also $(X_i)_{i=1,2}$ the dual vector field with respect to h of $(\alpha_i)_{i=1,2}$. Then remark that $X_i = x_i^1 \frac{\partial}{\partial x} + x_i^2 \frac{\partial}{\partial y}$ and $e_k = E_k^1 \frac{\partial}{\partial x} + E_k^2 \frac{\partial}{\partial y}$, where $(x_i^j)_{i,j=1,2}$ and (E_k^i) are functions from M to $\mathbb{R}$. Now write

$$-d^*\alpha_i = \sum_{1\leq k\leq n+2} h(\nabla_{e_k} X_i, e_k).$$

As the Levi-Civita connection is torsion free, we remark that for $i = 1, 2$ and $1 \le k \le n+2$

$$h(\nabla_{e_k} X_i, e_k) = h(\nabla_{X_i} e_k, e_k) + h([X_i, e_k], e_k).$$

However for $i = 1, 2$ and $1 \le k \le n+2$

$$h(\nabla_{X_i} e_k, e_k) = \frac{1}{2} X_i \cdot h(e_k, e_k) = 0$$

and noticing that for $1 \le k \le n$, $[\frac{\partial}{\partial x}, e_k] = [\frac{\partial}{\partial y}, e_k] = 0$ we obtain

$$[X_i, e_k] = [x_i^1 \tfrac{\partial}{\partial x} + x_i^2 \tfrac{\partial}{\partial y}, e_k] = -(e_k \cdot x_i^1) \tfrac{\partial}{\partial x} - (e_k \cdot x_i^2) \tfrac{\partial}{\partial y}.$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are in TT^2 we get that for all $1 \le k \le n$ and $i = 1, 2$,

$$h(\nabla_{e_k} X_i, e_k) = 0.$$

Now for $k = n+1, n+2$ we have

$$[X_i, e_k] = [x_i^1 \frac{\partial}{\partial x} + x_i^2 \frac{\partial}{\partial y}, E_k^1 \frac{\partial}{\partial x} + E_k^2 \frac{\partial}{\partial y}].$$

However, for any function f define on M, $TT^2 \in \ker df$ and noticing that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$, we finally get that α_i is indeed harmonic. □

Let b_1 be the first Betti number of our manifold M. Using an integral basis of harmonic 1-forms we can define, by integration, the Albanese (or Jacobi) map π; this gives a map onto a torus T^{b_1}, on which we put the usual flat metric. In our case, as all harmonic 1-forms are of constant length, we are allowed to choose this basis to be (pointwisely) orthonormal. Therefore π is a Riemannian submersion; besides we must have $b_1 \le n$. Moreover Albanese's map being harmonic (see A. Lichnerowicz [Lic69]; it is also a harmonic morphism in the sense of [BE81]), the fibers are minimal (see J. Eells and J.M. Sampson [ES64] and 9.34 of [Bes87], page 243).

A Theorem of R. Hermann (see Theorem 9.3 in [Bes87], page 235) states that a Riemannian submersion whose total space is complete is a locally trivial fibre bundle. We sum all this up in the following proposition.

Proposition 5. *Let (M^n, g) be a compact Riemannian manifold and b_1 be its first Betti number. Then all harmonic 1-forms are of constant length if and only if (M^n, g) is a locally trivial fiber bundle, with minimal fibers, over a b_1-dimensional flat torus, $b_1 \le n$*

$$F^{n-b_1} \hookrightarrow M^n \xrightarrow{\pi} T^{b_1}.$$

Moreover if $b_1 = n$, then π is a Riemannian isometry, hence (M^n, g) is a flat torus.

Now let us look at some other consequences of the existence of harmonic forms of constant length. Thanks to the Albanese map we can lift the harmonic forms of the Albanese torus on the manifold. Let us call $\alpha_1, \ldots, \alpha_{b_1}$ an orthonormal family of lifted harmonic forms. Using the duality through the metric we can associate to these harmonic forms b_1 vector fields $X_1, \ldots, X_{b_1}$. These vector fields define a sub-bundle $\mathcal{H}$, which we will call the **Horizontal**, such that for all $x \in M$, $\mathcal{H}_x$ is generated by $X_1(x), \ldots, X_{b_1}(x)$.

If we let $\mathcal{V}$ be the orthogonal complement of $\mathcal{H}$ with respect to the metric, which we will refer to as the **Vertical**, then we have the following.

Proposition 6. *Let (M^n, g) be a compact Riemannian manifold and let b_1 be its first Betti number. If all harmonic 1-forms are of constant length, then there is a distribution $\mathcal{H}$ given by an orthonormal frame of vector fields $X_1, \ldots, X_{b_1}$ dual to an orthonormal frame of harmonic 1-forms, such that the tangent bundle splits orthogonaly as follows:*

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

Moreover the distribution $\mathcal{V}$ is integrable, and for any $1 \le i, j \le b_1$ and $U \in \mathcal{V}$, $[X_i, X_j] \in \mathcal{V}$ and $[X_i, U] \in \mathcal{V}$.

Proof. This comes from the fact that the forms $(\alpha_i)_{1 \le i \le b_1}$ are closed. Indeed for any closed 1-form α we have the following equality for any X, Y in TM:

$$\alpha([X, Y]) = X.\alpha(Y) - Y.\alpha(X).$$

Thus for any i, j and k we have

$$\alpha_i([X_j, X_k]) = X_j.\alpha_i(X_k) - X_k.\alpha_i(X_j) = X_j \cdot \delta_{ik} - X_k.\delta_{ij} = 0,$$

hence $[X_j, X_k]$ is orthogonal to any X_i. If U and V are vertical vector fields, then it is easily seen that for any $1 \le i \le b_1$,

$$\alpha_i([X_j, U]) = 0 \quad \text{and} \quad \alpha_i([U, V]) = 0.$$

□

2.2. **A useful decomposition.** Let (M^n, g) be a compact Riemannian manifold with a unit vector field Z. Let $\mathcal{V}$ be the distribution generated by Z (which is sometimes called the **Vertical** distribution) and $\mathcal{H}$ its orthogonal complement with respect to g (which we will call **Horizontal**). Then the tangent bundle splits as follows:

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

If $i_Z(\cdot) = Z \lrcorner \cdot$ is the interior product by Z, then we can define the space of horizontal p-forms as

$$\Lambda^p(\mathcal{H}) = \Lambda^p(M) \cap \ker(i_Z).$$

Let us introduce the 1-form $\vartheta = Z^\flat$ dual to Z with respect to g. Then it is an easy exercise to see that we have the following decomposition of p-forms:

$$\Lambda^p(M) = \Lambda^p(\mathcal{H}) \oplus \Big[\Lambda^{p-1}(\mathcal{H}) \wedge \vartheta\Big]. \tag{2}$$

Notice that following this decomposition we have

$$d\vartheta = b + \eta \wedge \vartheta,$$

where b is a horizontal 2-form called the **curvature** of the horizontal distribution $\mathcal{H}$ and η is the horizontal one form associated to the horizontal vector field $-\nabla_Z Z$ with respect to g, a fact the reader may easily verify.

Let us introduce the horizontal exterior differential d_H, which associates to a horizontal differential form the horizontal part of its exterior differential. We also introduce the multiplication operators

$$\begin{aligned} L &: \Lambda^q(\mathcal{H}) \to \Lambda^{q+2}(\mathcal{H}), \quad L := \cdot \wedge b, \\ S &: \Lambda^q(\mathcal{H}) \to \Lambda^{q+1}(\mathcal{H}), \quad S := \eta \wedge \cdot. \end{aligned}$$

Thanks to the decomposition (2), each p-form may be identified to a couple $(\alpha, \beta) \in \Lambda^p(\mathcal{H}) \times \Lambda^{p-1}(\mathcal{H})$ of horizontal forms. Thus we can see the exterior differential d acting on p-forms as an operator from $\Lambda^p(\mathcal{H})\times\Lambda^{p-1}(\mathcal{H})$ to $\Lambda^{p+1}(\mathcal{H})\times\Lambda^{p+2}(\mathcal{H})$. Then, we have the following proposition, where $\mathcal{L}_Z$ is the Lie derivative in the direction of Z, and where for any differential operator D, D^* is its formal adjoint with respect to g.

Proposition 7 ([Nag01])**.** *With respect to the decomposition* (2) *we have for the exterior differential acting on p-forms:*

$$d = \begin{pmatrix} d_H & (-1)^{p-1}L \\ (-1)^p\mathcal{L}_Z & d_H + S \end{pmatrix},$$

and for the codifferential:

$$d^* = \begin{pmatrix} d_H^* & (-1)^{p-1}\mathcal{L}_Z^* \\ (-1)^p L^* & d_H^* + S^* \end{pmatrix}.$$

3. The case of the first Betti number being one less than the dimension

The aim of this section is to describe the $(n+1)$-dimensional orientable manifolds having all of their harmonic one-forms of constant length, and with first Betti number equal to n. All manifolds considered here are of dimension greater than or equal to 3 because in dimension 1 and 2 we have only the circle S^1 and the torus T^2 which admit forms of constant length.

We will now give a more detailed discussion of the bundle-like metrics appearing in the statement of Theorem 1. Start with an n-dimensional compact torus T^n, that is, a compact quotient of the form $\mathbb{R}^n/L$ where $L \subseteq \mathbb{R}^n$ is a co-compact lattice. Given a closed 2-form ω on T^n representing an integral cohomology class in $H^2_{DR}(T)$ one constructs a principal $\mathbb{S}^1$ bundle $\mathbb{S}^1 \hookrightarrow N \xrightarrow{\pi} T^n$ whose first Chern class equals $[\omega]$. At this stage we know that the manifold N has the structure of a 2-nilmanifold with 1-dimensional center (see [PS61]).

Now the manifold N has a particular class of Riemannian metrics that can be constructed as follows. Pick a flat Riemannian metric h on T^n, a principal connection 1-form ϑ on N and a positive smooth function f on T^n. On N we consider the Riemannian metric

$$g = \pi^\star h + f\circ\pi\cdot\vartheta\otimes\vartheta.$$

These metrics are all $\mathbb{S}^1$-invariant, and, in fact, the metrics g already constructed exhaust the class of $\mathbb{S}^1$-invariant metrics on N such that π is a Riemannian submersion to (T^n, h). Various properties of the metrics g can be read on the set (h, f, ω) of defining data. For example g is a left-invariant metric if and only if f is constant and ω is harmonic (with respect to h).

Definition 8. Let $\mathbb{S}^1 \hookrightarrow N^{n+1} \to T^n$ be a 2-step nilmanifold whose center is one dimensional. A **minimal bundle-like metric** on N is any Riemannian metric on N arising from the previously explained construction, with defining data (h, f, ω) such that f is a positive constant.

Thanks to this definition we can prove our characterization, which is recalled below.

Theorem 9. *Let* (N^{n+1}, g) *be a compact orientable connected manifold such that all of its harmonic* 1*-forms are of constant length, and such that* $b_1(N^{n+1}) = n$.

Then N^{n+1} is a 2-step nilmanifold whose kernel is one dimensional and g is a minimal bundle-like metric.

Proof. We deduce from Proposition 1 that we have the following fibration, with minimal fibers:

$$\mathbb{S}^1 \hookrightarrow N^{n+1} \xrightarrow{\pi} T^n,$$

where π is the Albanese map.

We begin by taking a basis $(a_1, \dots, a_n)$ of harmonic forms over the Albanese torus and lift it to a basis of harmonic 1-forms $(\alpha_1, \dots, \alpha_n)$ over N^{n+1}. Let $X_1, \dots, X_n$ be their dual vector fields with respect to the metric; they span, by definition, $\mathcal{H}$.

Now take Z the dual vector field to the 1-form $Z^\flat = *(\alpha_1 \wedge \cdots \wedge \alpha_n)$ ($*$ is the Hodge operator, thus this form is co-closed). Its length is constant by construction, and can be assumed to be 1 without loss of generality. Furthermore Z belongs to and spans $\mathcal{V}$.

Thanks to Proposition 6, for any i, j,

$$[Z, X_i], [X_i, X_j] \in \mathcal{V}. \tag{3}$$

Let us deduce some important facts from (3):

First, using the constant lentgh assumption we obtain the following sequence of equalities:

$$\begin{aligned} g(\nabla_{X_i} X_j, Z) = -g(X_j, \nabla_{X_i} Z) = \underbrace{-g(X_j, \nabla_Z X_i) = g(X_i, \nabla_Z X_j)}_{\text{by } (3)} \\ = g(X_i, \nabla_{X_j} Z) = -g(\nabla_{X_j} X_i, Z) \end{aligned}$$

which implies that

$$g([X_i, X_j], Z) = 2g(\nabla_{X_i} X_j, Z).$$

If we take X_k instead of Z we would obtain in the same way, for any $1 \leq k \leq n$,

$$0 = g([X_i, X_j], X_k) = 2g(\nabla_{X_i} X_j, X_k).$$

Hence for any $1 \leq i, j \leq n$ we finally get that

$$[X_i, X_j] = 2\nabla_{X_i} X_j \in \mathcal{V}. \tag{4}$$

Second, remark that for any co-closed one form α we have

$$\sum_k \big(\nabla_{X_k} \alpha\big) X_k + \big(\nabla_Z \alpha\big) Z = 0. \tag{5}$$

Taking for α each of the α_i in turn, since $[X_i, X_j] = 2\nabla_{X_i} X_j \in \mathcal{V}$, we deduce from equality (5), that for $i = 1, \dots, n$,

$$g(\nabla_Z X_i, Z) = 0.$$

However as the Levi-Civita is torsion free, for $i = 1, \dots, n$,

$$-g(X_i, \nabla_Z Z) = g(\nabla_Z X_i, Z) = g(\nabla_{X_i} Z, Z) + g([Z, X_i], Z) = g([Z, X_i], Z)$$

but $[Z, X_i] \in \mathcal{V}$. Hence for any $i = 1, \dots, n$,

$$[X_i, Z] = 0 \tag{6}$$

and $\nabla_Z Z \in \mathcal{V}$. However, Z being of constant norm we conclude that

$$\nabla_Z Z = 0. \tag{7}$$

Now from (7), (4) and (6) we easily get that Z is a Killing vector field.

Hence, from (3) we have the existence of functions f_{ij} on N such that

$$[X_i, X_j] = f_{ij} Z. \tag{8}$$

However we would like to have some structural constants instead of the functions f_{ij}. Using that Z is a Killing vector field we obtain for any $X, Y \in TN$,

$$dZ^\flat(X, Y) = 2g(\nabla_X Z, Y). \tag{9}$$

Thus, if we decompose $dZ^\flat$ in the basis given by $\alpha_i \wedge \alpha_j$ and $Z^\flat \wedge \alpha_i$ for all i, j, then, thanks to (7) and (9), we get that

$$dZ^\flat = \sum_{i<j} f_{ij} \alpha_i \wedge \alpha_j. \tag{10}$$

In other words $dZ^\flat$ is horizontal and as it can be easily verified that $\mathcal{L}_V(dZ^\flat) = O$ for any $V \in \mathcal{V}$, it is projectable, i.e., there exists a unique 2-form β on the Albanese torus such that $dZ^\flat = \pi^* \beta$. Remark that $d\beta = 0$, thus $\beta = \beta_0 + d\alpha$ by the Hodge-de Rham theorem, with $\alpha \in \Lambda^1(T^{b_1})$ and β_0 harmonic. Hence if $\zeta_0 = Z^\flat - \pi^* \alpha$, then $d\zeta_0 = \pi^* \beta_0$. But now $\beta_0 = \sum_{ij} c_{ij} a_i \wedge a_j$, where the c_{ij} are constants. This implies that

$$d\zeta_0 = \sum_{ij} c_{ij} \alpha_i \wedge \alpha_j. \tag{11}$$

We are now taking as a basis of vector fields the dual base $(X_1^0, \ldots, X_{n-1}^0, Z_0)$ of the base $(\alpha_1, \ldots, \alpha_{n-1}, \zeta_0)$ (i.e. $\alpha_i(X_j^0) = \delta_{ij}$, $\ker \zeta_0 = \langle X_1^0, \ldots, X_{n-1}^0 \rangle$ and $\zeta_0(Z_0) = 1$). Then we have from (11):

$$[X_i^0, Z_0] = 0,$$
$$[X_i^0, X_j^0] = c_{ij} \cdot Z_0.$$

Thus we can build a homomorphism I_A between the Lie algebra $\mathfrak{a}$ defined by $A_i, \ldots, A_n, A_{n+1}$ and with brackets

$$[A_i, A_j] = c_{ij} A_{n+1}$$

all the other brackets being equal to zero, by taking $I_A(A_i) = X_i^0$ for $i = 1, \ldots, n$ and $I_A(A_{n+1}) = Z_0$.

Now if $\mathcal{A}$ is the simply connected Lie group associated to $\mathfrak{a}$, thanks to the compactness of N^{n+1} we can integrate the homomorphism I_A to obtain an action of $\mathcal{A}$ on N^{n+1} (see corollaries 3 and 4 of theorem 2.9, page 113 in [Oni93]). Since each orbit is open and N^{n+1} is connected, this action is transitive. From this we deduce (see F.W. Warner [War83], Theorem, 3.62) that N^{n+1} is a Lie group. Thanks to the constant of structures (c_{ij}) we deduce that it is a two-step nilpotent Lie group (see M. Spivak [Spi79], volume I, Theorem 17 in chapter 10, for example) and we inherit from I_A that ζ_0 is a principal connection 1-form. Concerning g, it is minimal bundle-like by the above discussion. □

It is worth noticing that if the manifold is not orientable this is no longer true. Indeed counterexamples can be obtained by taking the product of a Klein bottle with a flat torus. A natural question arising is whether one can characterize the left invariant metrics among the minimal bundle-like metrics, in term of properties of harmonic forms. We first look at the 3-dimensional case.

Theorem 10. *Let (N^3, g) be a compact orientable connected manifold such that all harmonic 1-forms are of constant length, $b_1(N^3) = 2$, and such that the wedge product of two harmonic 1-forms is an eigenform of the laplacian. Then (N^3, g) is a compact quotient of the 3-dimensional Heisenberg group and g is a left-invariant metric.*

Proof. From the added assumption we get that $Z^\flat$ of the previous proof is an eigenform of the laplacian. That is to say, there is some constant λ such that

$$\Delta Z^\flat = \lambda Z^\flat.$$

However, using the useful decomposition, knowing that $Z^\flat$ is coclosed and $dZ^b = b = \pi^*(\beta)$ (following the decomposition (2), b corresponds to $(b, 0)$ and $Z^\flat$ corresponds to $(0, 1)$), we must also have that

$$\Delta Z^\flat = d^* dZ^b = d^* b = (L^* b) Z^\flat = (L^* L \cdot 1) Z^\flat = |\beta|^2 Z^\flat,$$

which means that β is of constant length. But β is a 2-form on a 2-dimensional torus, which also means that β is proportional to the volume form. In other words, the function (which is unique in that case) in the equality (10) is a constant. Remark that this also tells us that the eigenvalue is $\lambda = |\beta|^2 = f_{12}^2$. □

Now for the higher-dimensional case.

Theorem 11. *For $n > 2$, let (N^{n+1}, g) be a compact orientable connected manifold such that all harmonic 1-forms are of constant length, $b_1(N^{n+1}) = n$ and the wedge product of any $n-1$ harmonic 1-forms is an eigenform of the Laplacian. Then (N^{n+1}, g) is a 2-step nilmanifold whose center is one dimensional and g is a left invariant metric.*

Proof. We use the same notations as in the proof of Theorem 9. From the new assumption we get, thanks to the star Hodge operator, that for any harmonic 1-form α, $\alpha \wedge Z^\flat$ is a co-closed eigenform. Using the decomposition (2) we associate to $\alpha \wedge Z^\flat$ the pair $(0, \alpha)$. As $\alpha \wedge Z^\flat$ is coclosed $\Delta(\alpha \wedge Z^\flat) = d^* d(\alpha \wedge Z^\flat)$. Let us use Proposition 7 without forgetting that α is harmonic, and noticing that as $dZ^\flat$ is horizontal by the proof of Theorem 9, $S = 0$

$$d(\alpha \wedge Z^\flat) \equiv \begin{pmatrix} d_H & -L \\ \mathcal{L}_Z & d_H \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} -L\alpha \\ d_H \alpha \end{pmatrix} = \begin{pmatrix} -L(\alpha) \\ 0 \end{pmatrix}$$

and

$$d^* d(\alpha \wedge Z^\flat) \equiv \begin{pmatrix} d_H^* & \mathcal{L}_Z^* \\ -L^* & d_H^* \end{pmatrix} \begin{pmatrix} -L(\alpha) \\ 0 \end{pmatrix} = \begin{pmatrix} d_H^*\big(-L(\alpha)\big) \\ L^* L(\alpha) \end{pmatrix},$$

but the last term is also equal to $(0, \lambda\alpha)$ by the eigenform assumption, hence

$$d_H^*(\alpha \wedge dZ^\flat) = 0.$$

But α and $dZ^\flat$ are horizontal and projectable, thus in fact there is a 1-form a and a 2-form β on the torus such that $\alpha = \pi^* a$, $dZ^\flat = \pi^* \beta$ and we can write

$$0 = d_H^*(\alpha \wedge dZ^\flat) = \pi^*\big(d^*(a \wedge \beta)\big).$$

In other words, for any 1-harmonic form a on the torus we have

$$d^\star(a \wedge \beta) = 0. \tag{12}$$

Hence we are led to work on the torus.

Let us take an orthonormal base (e_i) of parallel vector fields over the torus. Then for any form ω,

$$-d^\star\omega = \sum_i e_i \lrcorner \nabla_{e_i}\omega.$$

We want to apply that last equality to $a \wedge \beta$. First notice that (a is parallel)

$$\nabla_{e_i}(a \wedge \beta) = a \wedge \nabla_{e_i}\beta.$$

Contracting by e_i one obtains

$$e_i \lrcorner \big(\nabla_{e_i}(a \wedge \beta)\big) = a(e_i)\nabla_{e_i}\beta - \alpha \wedge (e_i \lrcorner \nabla_{e_i}\beta).$$

Sum over i and use the coclosed condition (i.e. (12)) and you get

$$\nabla_{a^\#}\beta + a \wedge d^\star\beta = 0. \tag{13}$$

Now we take the interior product with e_i of (13), with $a = e_i^\flat$, which gives

$$e_i \lrcorner \nabla_{e_i}\beta + e_i \lrcorner (e_i^\flat \wedge d^\star\beta) = e_i \lrcorner \nabla_{e_i}\beta + d^\star\beta - d^\star\beta(e_i)e_i^\flat = 0.$$

We sum over i one more time to obtain

$$(n-2)d^*\beta = 0$$

and by assumption $n > 2$, thus $d^*\beta = 0$; that is, β is harmonic over the torus. Hence it follows that β is parallel and all the functions f_{ij} of (10) are constants, which allows us to conclude. □

The Heisenberg groups being a model space of contact geometry, it is a natural setting to study under our assumptions. In the contact case we obtain

Theorem 12. *Let $(N^{2m+1}, \omega, g_\omega)$ be a compact contact manifold with a contact form ω and an adapted Riemannian metric g_ω such that all harmonic 1-forms are of constant length. Then (N^{2m+1}, g) is a compact quotient of a Heisenberg group and g is a left invariant metric.*

Proof. From Theorem 1 we get that N^{2m+1} is a two-step nilmanifold and g_ω is minimal bundle-like. We also get that $g_\omega = \vartheta^2 + \pi^*(h)$, where ϑ is a one form, such that $d\vartheta = \pi^*(\beta)$ for some closed 2-form β over the Albanese torus. Now Theorem 3.2 and Theorem 3.4 of H.P. Pak and T. Takahashi in [PT01] imply that for all harmonic 1-forms α, if T is the Reeb vector field attached to ω,

$$T \lrcorner \alpha = \alpha(T) = 0,$$

which means that $T = fZ$ for some function f, and as T and Z are of constant unit length for the metric g_ω it means that $\omega = \vartheta$. Now, thanks to Theorem 1 we know that $T = Z$ is a Killing field. This implies that the almost-complex structure J on $\ker\omega$ lives on the flat torus given by the Albanese submersion. Hence we are in front of an almost-Kähler flat torus, but following [Ols78] and [Arm02], it has to be Kähler, thus β is parallel. □

4. Some remarks on the general case

The aim of this section is to point out the main differences between the case $b_1 = n-1$ and $b_1 \leq n-2$ for n-dimensional manifolds admitting one-harmonic forms of constant length. We want to give some hint on the failure of our approach.

Our first remark is that one should restrict oneself to the study of locally irreducible orientable Riemannian manifolds to avoid the following cases: the direct product of a sphere of dimension $p > 1$ and a flat torus of dimension $n-p$ gives a manifold whose first Betti number is $n-p$, whose dimension is n and with $n-p$ harmonic 1-forms of constant length.

The second remark is in the following lemma, which shows the limitation of our method. Indeed to apply the same ideas one needs far stronger assumptions.

Lemma 13. *Let (N^n, g) be a compact locally irreducible manifold such that all harmonic 1-forms are of constant length, $b_1(N^n) = n-p$ and possessing a pointwise orthonormal base (ϑ_i) of the orthogonal complement of the harmonic 1-forms. Moreover assume that $(d\vartheta_i)$ are lifts of closed 2-forms on the Albanese torus. Then (N^n, g) is a two-step nilpotent nilmanifold whose kernel is $n-p$ dimensional.*

Proof. $TM = \mathcal{V} + \mathcal{H}$, where $\mathcal{H}$ is spanned by $X_1, \ldots, X_{n-p}$ the dual vector fields to $\alpha_1, \ldots, \alpha_{n-p}$, which are lifts of harmonic 1-forms on the Albanese torus, and $\mathcal{V}$ is the orthogonal complement. We associate, thanks to the metric, the dual vector fields (Z_k) to the 1-forms (ϑ_k). As the α_i are closed we get that for $1 \leq i, j \leq n-p$,

$$[X_i, X_j] \in \mathcal{V}.$$

Our assumptions imply that

$$d\vartheta_k \in \Lambda^2 \mathcal{H},$$

where $\Lambda^2\mathcal{H} = \Lambda^2 M \cap \bigcap_k \ker i_{Z_k}$ and that

$$d\vartheta_k = \pi^\star(\beta_k),$$

where β_k is a 2-form on Albanese's torus. However, $d\beta_k = 0$, hence for some harmonic 2-form β_k^0 and some 1-form a_k over the Albanese torus we have

$$\beta_k = \beta_k^0 + da_k.$$

Notice, that β_k^0 is non-zero; otherwise ϑ_k would be horizontal, which is not the case by assumption. Hence if we consider the independent forms (as one can easily verify)

$$\theta_k = \vartheta_k - \pi^*(a_k),$$

then

$$d\theta_k = \pi^*(\beta_k^0).$$

Hence we can conclude as in the proof of Theorem 1. $\square$

As the last section involved nilmanifolds, we could decide to focus on a family of compact manifolds close to them, say solvmanifold. But even that assumption is not enough to clarify the situation, as the next lemma with the following remark points out.

Lemma 14. *Let (M^n, g) be a solvmanifold all of whose 1-forms are of constant length and whose first Betti number is $n-2$. Then we have the following fibration with minimal fibers:*

$$T^2 \hookrightarrow M^n \xrightarrow{\pi} T^{n-2}.$$

Proof. This comes from the fact that the fibration in Proposition 5,

$$F^2 \hookrightarrow M^n \xrightarrow{\pi} T^{n-2},$$

gives a long exact sequence on the homotopy groups

$$\cdots \to \pi_n(F^2) \to \pi_n(M^n) \to \pi_n(T^{n-2}) \to \pi_{n-1}(F^2) \to \pi_{n-1}(M^n) \to \ldots.$$

Now, T^{n-2} and M^n have only their fundamental group which is not trivial, hence we have the following exact sequence on the fundamental groups:

$$0 \to \pi_1(F^2) \to \pi_1(M^n) \to \pi_1(T^{n-2}) \to 0,$$

and $\pi_k(F^2)$ is trivial if $k > 1$. This means that $\pi_1(F^2)$ can be seen as a subgroup of the solvable group $\pi_1(M^n)$, hence it is also solvable. Thus the fiber is compact and with solvable fundamental group. However in dimension 2 the only compact oriented manifold with solvable fundamental groups are the sphere and the torus, but here the sphere is excluded because $\pi_2(S^2) \neq 0$. $\square$

Without further assumptions we cannot expect a more precise result. Indeed the example of the Sol geometry in dimension 3, or of any 2-step nilmanifold with a 2-dimensional center will satisfy Lemma 14 with many different metrics, however built in the same way, following the construction 8.1 in [BK03b].

As a conclusion, the rigidity of the cases $b_1 = n$ and $b_1 = n - 1$ do not propagate to lower values of the first Betti number.

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