Axiomatic Symmetric Cones and the Thompson metric

Jimmie Lawson

Work with Yongdo Lim

Louisiana State University

Background

Let $\mathbf{A}$ be a unital C^*-algebra with identity e, and let $\mathbf{A}^+$ be the set of positive invertible elements of $\mathbf{A}$. It is known that $\mathbf{A}^+$ is an open convex cone in the space $H(\mathbf{A})$ of hermitian elements. The geometry of $\mathbf{A}^+$ has been studied by several authors. One particular focus in this geometry has been the study of appropriate non-positive curvature properties.

Background

Let $\mathbf{A}$ be a unital C^*-algebra with identity e, and let $\mathbf{A}^+$ be the set of positive invertible elements of $\mathbf{A}$. It is known that $\mathbf{A}^+$ is an open convex cone in the space $H(\mathbf{A})$ of hermitian elements. The geometry of $\mathbf{A}^+$ has been studied by several authors. One particular focus in this geometry has been the study of appropriate non-positive curvature properties.

One approach has been to endow $\mathbf{A}^+$ with a natural Finsler structure and metric and use this for a substitute for the Riemannian geometry commonly considered in finite-dimensional examples. For example, Andruchow-Corach-Stojanoff and Corach-Porta-Recht have shown the convexity of the distance function along two distinct geodesics and its equivalence to the well-known Loewner-Heinz inequality.

Background II

K.-H. Neeb established an appropriate differential geometric notion of seminegative (equal non-positive) curvature for certain classes of Finsler manifolds, particularly the infinite-dimensional symmetric spaces, including the case of $\mathbf{A}^+$.

Background II

K.-H. Neeb established an appropriate differential geometric notion of seminegative (equal non-positive) curvature for certain classes of Finsler manifolds, particularly the infinite-dimensional symmetric spaces, including the case of $\mathbf{A}^+$.

Our approach has been somewhat different from either of the preceding. We replace the differential geometric structure by the structure of a *metric* symmetric space endowed with a midpoint operation and study seminegativity via the metric. We have obtained the convexity of the metric for such symmetric spaces with weaker metric assumptions than those enjoyed by the Finsler metric on $\mathbf{A}^+$.

Overview

The main goal of this presentation is to present a very general framework, that of a general notion of a symmetric cone, in which one can derive and study various inequalities, such as those familiar from operator theory.

Overview

The main goal of this presentation is to present a very general framework, that of a general notion of a symmetric cone, in which one can derive and study various inequalities, such as those familiar from operator theory.

Our notion of a symmetric cone is that of an open normal cone in a Banach space equipped with an axiomatic symmetric structure that is appropriately connected to the conal structure. All of this of course, needs to be made precise.

Proper and Normal Cones

Let V be a Banach space and let Ω henceforth denote a non-empty *proper* open convex cone of V: $t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω.

Proper and Normal Cones

Let V be a Banach space and let Ω henceforth denote a non-empty *proper* open convex cone of V: $t\Omega \subset \Omega$ for all $t > 0$, $\Omega + \Omega \subset \Omega$, and $\overline{\Omega} \cap -\overline{\Omega} = \{0\}$, where $\overline{\Omega}$ denotes the closure of Ω.

We further assume that Ω is a *normal cone*: there exists a constant K with $||x|| \leq K||y||$ for all $x, y \in \overline{\Omega}$ with $x \leq y$.

The Order

For a normal cone Ω, the relation

$$x \leq y \ \text{ if and only if } y - x \in \overline{\Omega}$$

is a partial order. We write $x < y$ if $y - x \in \Omega$.

The Order

For a normal cone Ω, the relation

$$x \leq y \;\text{ if and only if } y - x \in \overline{\Omega}$$

is a partial order. We write $x < y$ if $y - x \in \Omega$.

Any member ε of Ω is an order unit for the ordered space $(V, \leq)$, and the cone is normal if and only if the *order unit norm* determined by ε

$$\|x\|_\varepsilon = \inf\{M > 0 : -M\varepsilon \leq x \leq M\varepsilon\}$$

is compatible, i.e., determines the topology of V. In this case $0 \leq x \leq y$ implies $\|x\|_\varepsilon \leq \|y\|_\varepsilon$, that is, we may (and do) assume without loss of generality that $K = 1$.

The Thompson Metric

A. C. Thompson has shown that Ω is a complete metric space with respect to the *Thompson (part) metric*

$$d(x,y) = \max\{\log M(x/y), \log M(y/x)\},$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\}$.

The Thompson Metric

A. C. Thompson has shown that Ω is a complete metric space with respect to the *Thompson (part) metric*

$$d(x, y) = \max\{\log M(x/y), \log M(y/x)\},$$

where $M(x/y) := \inf\{\lambda > 0 : x \leq \lambda y\}.$

The Thompson metric can be alternatively realized as an appropriately defined *Finsler length metric* (Nussbaum): for $x \in \Omega$ and $v \in V = T_x\Omega$, $|v|_x (= \inf\{t > 0 : -tx \leq v \leq tx\})$. The Thompson part metric $d(x, y)$ agrees with the Finsler distance from x to y:

$$d(x, y) = \inf\{\int_0^1 |\gamma'(t)|_{\gamma(t)} : \gamma \in S, \gamma(0) = x, \gamma(1) = y\},$$

where $S =$the set of piecewise C^1-maps $\gamma : [0, 1] \to \Omega$.

Strict Symmetric Sets

A *strict symmetric set* is a triple $(X, \bullet, \#)$ with left translation $S_x y := x \bullet y$ representing the point symmetry through x, satisfying for all $a, b, c \in X$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_X)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \qquad (S_a S_b = S_{S_a b} S_a)$;

(S4) (strictness) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x = a \# b$ called the *midpoint of symmetry* or *mean* of a and b.

Strict Symmetric Sets

A *strict symmetric set* is a triple $(X, \bullet, \#)$ with left translation $S_x y := x \bullet y$ representing the point symmetry through x, satisfying for all $a, b, c \in X$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_X)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(S4) (strictness) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x = a\#b$ called the *midpoint of symmetry* or *mean* of a and b.

The binary operations $\bullet$ and $\#$ uniquely determine each other via

$$c = a\#b \Leftrightarrow c \bullet a = b.$$

The Loos Axioms

The Loos axioms for a symmetric space posit a topological (smooth) space X equipped with a continuous (smooth) binary operation $\bullet$ satisfying (S1)-(S3) and
(S4') For each $x \in X$, the fixed points of S_x are isolated.

Our variant (S4) implies (S4') since under (S4) the only fixed point of S_x is x itself. Thus the axioms for a strict symmetric space have a purely algebraic formulation.

Twisted Subgroups

A *twisted subgroup* P of a group G is a subset containing the identity e, closed under inversion, and under $(a, b) \mapsto aba$. With respect to $a \bullet b := ab^{-1}a$ (sometimes called the “core" operation), P satisfies (S1), (S2), and (S3).

Twisted Subgroups

A *twisted subgroup* P of a group G is a subset containing the identity e, closed under inversion, and under $(a,b) \mapsto aba$. With respect to $a \bullet b := ab^{-1}a$ (sometimes called the “core" operation), P satisfies (S1), (S2), and (S3).

The twisted subgroup P is a strict symmetric set if and only if it is uniquely 2-divisible (every element of P has a unique square root in P). In this case $a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}$.

Twisted Subgroups

A *twisted subgroup* P of a group G is a subset containing the identity e, closed under inversion, and under $(a,b) \mapsto aba$. With respect to $a \bullet b := ab^{-1}a$ (sometimes called the "core" operation), P satisfies (S1), (S2), and (S3).

The twisted subgroup P is a strict symmetric set if and only if it is uniquely 2-divisible (every element of P has a unique square root in P). In this case $a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}$.

Example. For a C^*-algebra $\mathbf{A}$, the positive part $\mathbf{A}^+ = \{xx^* : x \in \mathbf{A}\} \cap G(\mathbf{A})$ is such a twisted subgroup of the group $G(\mathbf{A})$ of invertible elements.

A Converse

Let ε be a distinguished point of a strict symmetric set $(P, \bullet, \#)$. For $x \in P$, define the *displacement* $Q(x) : P \to P$ by $Q(x) = S_x S_\varepsilon$. The displacements are isomorphisms of P and generate the *displacement group* $G(P)$.

A Converse

Let ε be a distinguished point of a strict symmetric set $(P, \bullet, \#)$. For $x \in P$, define the *displacement* $Q(x) : P \to P$ by $Q(x) = S_x S_\varepsilon$. The displacements are isomorphisms of P and generate the *displacement group* $G(P)$.

The map $x \mapsto Q(x)$ from P to $G(P)$, called the *quadratic representation*, carries P to $Q(P)$, a uniquely 2-divisible twisted subgroup of $G(P)$, and is an isomorphism onto $Q(P)$ endowed with the core operation and corresponding mean operation.

Weighted Means

The core of the group $(\mathbb{R}, +)$ has operations

$$\lambda \bullet \mu = \lambda - \mu + \lambda = 2\lambda - \mu; \quad \lambda \# \mu = \frac{\lambda + \mu}{2},$$

and these operations restrict to a strict symmetric structure on the dyadic rationals $\mathbb{D}$.

Weighted Means

The core of the group $(\mathbb{R}, +)$ has operations

$$\lambda \bullet \mu = \lambda - \mu + \lambda = 2\lambda - \mu; \quad \lambda \# \mu = \frac{\lambda + \mu}{2},$$

and these operations restrict to a strict symmetric structure on the dyadic rationals $\mathbb{D}$.

Given $x \neq y$ in a strict symmetric set P, there exists a unique morphism of $(\mathbb{D}, \bullet, \#)$ to P taking 0 to x and 1 to y. We denote the image of $t \in \mathbb{D}$ by $x\#_t y$, called the t-*weighted mean* of x and y. Note $x\#_{1/2}y = x\#y$.

Powers

Dyadic powers for $(P, \bullet, \#, \varepsilon)$ arise as a special case of weighted means: $x^t = \varepsilon \#_t x$ for $x \in P$, $t \in \mathbb{D}$.

In the case of a uniquely 2-divisible twisted subgroup equipped with the core operation and with ε the identity, this notion agrees with powers in the group. Thus powers may be alternatively defined through the quadratic representation.

Powers

Dyadic powers for $(P, \bullet, \#, \varepsilon)$ arise as a special case of weighted means: $x^t = \varepsilon \#_t x$ for $x \in P$, $t \in \mathbb{D}$.

In the case of a uniquely 2-divisible twisted subgroup equipped with the core operation and with ε the identity, this notion agrees with powers in the group. Thus powers may be alternatively defined through the quadratic representation.

For $s, t \in \mathbb{D}$, $x \in P$, exponents satisfy the laws

$$x^{st} = x^{st}, \quad x^s \bullet x^t = x^{2s-t}, \quad x^s \# x^t = x^{(s+t)/2}.$$

Symmetric Spaces

A *pointed strict symmetric space with convex metric* is a pointed strict symmetric set P equipped with a complete metric $d(\cdot,\cdot)$ satisfying for all $x, y \in P$ and $g \in G(P)$

(i) $d(g.x, g.y) = d(x, y)$,

(ii) $d(x^{-1}, y^{-1}) = d(x, y)$,

(iii) $d(\varepsilon, x^{1/2}) = (1/2)d(\varepsilon, x)$,

(iv) $d(x^{1/2}, y^{1/2}) \leq \frac{1}{2}d(x, y)$ (Busemann nonpositive curvature),

(v) $x \mapsto x^2 : P \to P$ is continuous.

A *strict symmetric space with convex metric* is a strict symmetric set equipped with a complete metric that is a pointed strict symmetric space with convex metric with respect to some pointing.

Symmetric Space Properties

Let $(P, \bullet, \#)$ be a pointed strict symmetric space with convex metric.

1. For $x \neq y$, $\exists!\ \alpha_{x,y} : (\mathbb{R}, \bullet, \#) \to (P, \bullet, \#)$ a continuous homomorphism that is also a metric geodesic such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$.
2. For every pair (β, γ) of such geodesics, the real function
$$t \mapsto d(\beta(t), \gamma(t))$$
is a convex function (Busemann NPC II).
3. The maps $(x, y) \mapsto x \bullet y : P \times P \to P$ and $(t, x, y) \mapsto \alpha_{x,y}(t) := x\#_t y : \mathbb{R} \times P \times P \to P$ are continuous. The element $x\#_t y$ is called the t-*weighted mean* of x and y. Note that $x\#y = x\#_{1/2}y$.

Cones With Symmetric Structures

Key Lemma. Let Ω be an open normal cone in a Banach space V. Suppose in addition that $(\Omega, \bullet, \#, \varepsilon)$ is a pointed strict symmetric set such that the displacements $Q(x) : \Omega \to \Omega$ are positively homogeneous for all $x \in \Omega$. Then $(\lambda x)^{-1} = \frac{1}{\lambda} x^{-1}$ and $\lambda^t \varepsilon = (\lambda \varepsilon)^t$ for all $\lambda > 0$ and all dyadic rationals t. Moreover $\mu x \#_t \lambda x = \mu^{1-t} \lambda^t x$ for all $x \in \Omega$, $\mu, \lambda > 0$ and all dyadic rationals t. Furthermore, the following conditions are equivalent:

(i) $(\lambda x)^{1/2} = \sqrt{\lambda} x^{1/2}$ for all $x \in \Omega$ and $\lambda > 0$;

(ii) $(\lambda x)^t = \lambda^t x^t$ for any dyadic rational t and $x \in \Omega$ and $\lambda > 0$;

(iii) $Q(\lambda x) = \lambda^2 Q(x)$ for any $x \in \Omega$ and $\lambda > 0$.

Key Theorem I

Theorem. Let Ω be an open normal cone in a Banach space V. Suppose that there is a pointed strict symmetric structure on Ω satisfying

(i) $\varepsilon\#x = x^{1/2} \leq (\varepsilon + x)/2$.

(ii) the squaring map $x \mapsto x^2 = Q(x)\varepsilon$ is continuous $($in the relative norm topology of $\Omega)$.

(iii) every basic displacement $Q(x)$ is continuous and linear $($that is, additive and positively homogeneous$)$ on Ω.

Then Ω is a strict symmetric space with convex metric with respect to the Thompson metric that satisfies the equivalent conditions (i), (ii) and (iii) of the Key Lemma.

Key Theorem II

Furthermore, (i) the order-reversing property of inversion, (ii) the harmonic-geometric-arithmetic mean inequality, and (iii) the Loewner-Heinz inequality all hold: for $a, b \in \Omega$,

(i) $b^{-1} \leq a^{-1}$ if $a \leq b$,

(ii) $2(a^{-1} + b^{-1})^{-1} \leq a\#b \leq \frac{1}{2}(a+b)$, and

(iii) $a^t \leq b^t$ if $a \leq b$, $0 \leq t \leq 1$.

Key Theorem II

Furthermore, (i) the order-reversing property of inversion, (ii) the harmonic-geometric-arithmetic mean inequality, and (iii) the Loewner-Heinz inequality all hold: for $a, b \in \Omega$,

(i) $b^{-1} \leq a^{-1}$ if $a \leq b$,

(ii) $2(a^{-1} + b^{-1})^{-1} \leq a\#b \leq \frac{1}{2}(a+b)$, and

(iii) $a^t \leq b^t$ if $a \leq b$, $0 \leq t \leq 1$.

Cones satisfying the hypotheses (and hence conclusions) of the preceding theorem will be called *Loos symmetric cones*.

Nonpositive Curvature

It is interesting to note that in the context of cones equipped with symmetric structures the weakened form of the geometric-arithmetic mean inequality

$$\varepsilon \# x = x^{1/2} \leq (\varepsilon + x)/2$$

yields both the Loewner-Heinz inequality and Busemann nonnegative curvature (in the sense that the Thompson metric is convex).

Question: Is convexity of the Thompson metric equivalent to convexity of the Hilbert metric?

Example: C^*-Algebras

If $\mathbb{A}$ is a unital C^*-algebra with identity e, then the cone $\mathbb{A}^+$ of positive invertible elements is an open normal convex cone in the closed subspace $\mathcal{H}(\mathbf{A})$ of hermitian elements. The set $\mathbb{A}^+$ is a twisted subgroup (closed under $(x, y) \mapsto xy^{-1}x$) with unique square roots of the multiplicative group of invertible elements of $\mathbb{A}$, hence a pointed strict symmetric set with respect to $x \bullet y \ = xy^{-1}x$ and distinguished point the identity e. Furthermore, the powers computed in the algebra agree with those computed in $(\mathbb{A}^+, \bullet, e)$, so that the squaring map is continuous. The condition $x^{1/2} \leq (e + x)/2$ is equivalent to $(e - x)^2 \geq 0$, thus valid. Since $Q(x)y = x(y^{-1})^{-1}x = xyx$, the mapping $Q(x)$ is a continuous and linear on $\mathcal{H}(\mathbf{A})$.

It follows that the cone $\mathbb{A}^+$ is a Loos symmetric cone.

Harmonic-Arithmetic Approximation

Many results concerning the cone of positive elements in an algebra generalize to Loos symmetric cones.

For x, y in a Loos symmetric cone Ω in a Banach space V, define $H_1 = H(x, y)$, the harmonic mean, and $A_1 = A(x, y)$, the arithmetic mean. Inductively define $H_{n+1} = H(H_n, A_n)$ and $A_{n+1} = A(H_n, A_n)$. Then for each n,

$$H_n \leq H_{n+1} \leq x\#y \leq A_{n+1} \leq A_n,$$

and $H_n \to x\#y$, $A_n \to x\#y$.

Inequalities

Numerous operator inequalities remain valid in the more general context of Loos symmetric cones, for example,

The Furuta Inequality: Let Ω be a Loos symmetric cone in a Banach space V and let $0 < b \leq a$. If $0 \leq p, q, r \in \mathbb{R}$ satisfies $p + 2r \leq (1 + 2r)q$ and $1 \leq q$, then

$$b^{\frac{p+2r}{q}} \leq (b^r.a^p)^{\frac{1}{q}}.$$

JB-Algebras

A Jordan algebra is a vector space Z with a commutative multiplication xy such that $x(x^2y) = x^2(xy)$ holds for $x, y \in Z$. A JB-algebra V is a real Jordan algebra with unit e endowed with a complete norm $||\cdot||$ such that

$$||zw|| \ \leq \ ||z||\,||w||, \ \ ||z^2|| = ||z||^2, \ \ ||z||^2 \leq ||z^2 + w^2||.$$

(The hermitian elements $x = x^*$ of any C^*-algebra form a JB-algebra with respect to the symmetric product $x \circ y := (xy + yx)/2$.)

JB-Algebras

A Jordan algebra is a vector space Z with a commutative multiplication xy such that $x(x^2y) = x^2(xy)$ holds for $x, y \in Z$. A JB-algebra V is a real Jordan algebra with unit e endowed with a complete norm $||\cdot||$ such that

$$||zw|| \;\le\; ||z||\,||w||, \;\; ||z^2|| = ||z||^2, \;\; ||z||^2 \le ||z^2 + w^2||.$$

(The hermitian elements $x = x^*$ of any C^*-algebra form a JB-algebra with respect to the symmetric product $x \circ y := (xy + yx)/2$.)

Theorem. Let V be a JB-algebra and let $\Omega = \exp V$ be the associated positive cone. Then Ω is a Loos symmetric cone.

Hermitian Banach Algebras

Let Z be a unital Banach algebra Z with a continuous involution $*$ and let X consist of the self-adjoint elements of Z. Let e denote the unit element of Z. The unital Banach algebra Z is called *hermitian* if $\sigma(x) \subset \mathbb{R}$ and $||x|| = \sup|\sigma(x)|$ for every $x = x^*$. We let

$$\Omega := \{x = x^* : \sigma(x) \subset (0, \infty)\}.$$

Theorem. Let Z be a hermitian Banach algebra. Then Ω is a Loos symmetric cone of X.

Open Problem

The least squares mean A of positive matrices $A_1, \ldots, A_n$ minimizes

$$F(X) = \sum_{i=1}^{n} d^2(X, A_i),$$

where d is the trace Riemannian metric.

Problem. Does the least squares mean generalize to Loos symmetric cones?