Approximability of convex bodies and volume entropy in Hilbert geometry

Constantin Vernicos

Abstract

The approximability of a convex body is a number which measures the difficulty in approximating that convex body by polytopes. In the interior of a convex body one can define its Hilbert geometry. We prove on the one hand that the volume entropy is twice the approximability for a Hilbert geometry in dimension two end three, and on the other hand that in higher dimensions it is a lower bound of the entropy. As a corollary we solve the volume entropy upper bound conjecture in dimension three and give a new proof in dimension two from the one given in [BBV10]. Moreover, our method allows us to prove the existence of Hilbert geometries with intermediate volume growth one the one hand, and that in general the volume entropy is not a limit on the other hand.

Introduction and statement of results

Hilbert geometries are all the metric spaces obtained by defining the so-called Hilbert distance on open bounded convex sets in $\mathbb{R}^n$. The definition of this distance uses cross-ratios in the same way as in Klein projective model of the hyperbolic geometry [Hil71]. These metric spaces are actually length space whose structure is defined by a Finsler metric which is Riemannian if and only if the underlying open bounded convex set is an ellipsoid [Kay67].

These geometries have attracted a lot of interest see for example the works of Y. Nasu [Nas61], W. Goldmann [Gol90], P. de la Harpe [dlH93], A. Karlsson and G. Noskov [KN02], Y. Benoist [Ben03, Ben06], T. Foertsch and A. Karlsson [FK05], I. Kim [Kim05], B. Colbois, C. Vernicos and P. Verovic [CVV04, CVV06], B. Lins and R. Nussbaum [LN08], A. Borisenko and E. Olin [BO08, BO11], B. Lemmens and C. Walsh [LW11], C. Vernicos [Ver09, Ver11, Ver13], L. Marquis [Mar12], M. Crampon and L. Marquis [CM14], D. Cooper, D. Long and S. Tillman [CLT]), X. Nie [Nie] and the Handbook of Hilbert geometry [Hbk14].

The present paper focuses on the volume growth of these geometries and more specifically on the volume entropy.

Let Ω be a bounded open convex set in $\mathbb{R}$ endowed with its Hilbert geometry. If we consider the *Busemann* volume Vol_Ω and denote by $B_\Omega(p,r)$ the metric ball of radius r centred at the point $p\in\Omega$, then the *lower and upper volume entropies* of Ω will be defined respectively by

$$\underline{\mathrm{Ent}}(\Omega) = \liminf_{r\to+\infty} \frac{\ln\big(\mathrm{Vol}_\Omega(B_\Omega(p,r)\big)}{r}, \text{ and } \quad \overline{\mathrm{Ent}}(\Omega) = \limsup_{r\to+\infty} \frac{\ln\big(\mathrm{Vol}_\Omega(B_\Omega(p,r)\big)}{r}. \qquad (1)$$

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When the two limits coincide we denote their common limit by $\mathrm{Ent}(\Omega)$ and call it the *volume entropy* of Ω.

Let us stress out that in this definition the upper and lower volume entropy of Ω do not depend on the base point p and are actually projective invariants attached to Ω.

The question we address in this essay is twofold. On the one hand it is an investigation of the existence of an analogue, for all Hilbert geometries, of the relation between the volume entropy and the Hausdorff dimension of the radial limit set on the universal cover of a compact riemannian manifold with nonpositive curvature. On the other hand we focus on the *volume entropy upper bound conjecture* which states that if Ω is an open and bounded convex subset of $\mathbb{R}^n$, then $\overline{\mathrm{Ent}}(\Omega) \leqslant n-1$. To put our work into perspective let us recall the main related results.

The first one is a complete answer to the conjecture in the two-dimensional case by G. Berck, A. Bernig and C. Vernicos in [BBV10], where the authors actually obtained an upper bound as a function of d, the upper Minkowski dimension (or *ball-box* dimension) of the set of extreme points of Ω, naimely

$$\overline{\mathrm{Ent}}(\Omega) \leqslant \frac{2}{3-d} \leqslant 1. \tag{2}$$

The second result is a more precise statement with respect to the asymptotic volume growth of balls. It involves another projective invariant introduced by G. Berck, A. Bernig and C. Vernicos in the introduction of [BBV10] called the *centro-projective* area of Ω, and defined by

$$\mathcal{A}_p(\Omega) := \int_{\partial\Omega} \frac{\sqrt{k(x)}}{\langle n(x), x-p\rangle^{\frac{n-1}{2}}} \left(\frac{2a(x)}{1+a(x)}\right)^{\frac{n-1}{2}} \, dA(x), \tag{3}$$

where for any $x \in \partial\Omega$, $k(x)$ is the Gauss curvature, $n(x)$ the outward normal and $a(x) > 0$ the antipodal map defined by $p - a(x)(x-p) \in \partial\Omega$. Let us recall here that both k and n are defined almost everywhere as Alexandroff's theorem states [Ale39].

Now, the Second Main Theorem of G. Berck, A. Bernig and C. Vernicos in [BBV10] — which encloses former results given by B. Colbois and P. Verovic in [CV04] — asserts that in case $\partial\Omega$ is $C^{1,1}$ we have

$$\lim_{r\to+\infty} \frac{\mathrm{Vol}_\Omega B_\Omega(p,r)}{\sinh^{n-1} r} = \frac{1}{n-1}\mathcal{A}_p(\Omega) \neq 0 \tag{4}$$

and $\mathrm{Ent}(\Omega) = n-1$ is a limit. Moreover, without any assumption on Ω we have $\underline{\mathrm{Ent}}(\Omega) \geqslant n-1$ whenever $\mathcal{A}_p(\Omega) \neq 0$.

The third one — which is also a rigidity result — requires stronger assumptions about Ω: it has to be divisible, meaning that it admits a compact quotient, and its Hilbert metric has to be hyperbolic in the sense of Gromov, which implies its boundary is C^1 and strictly convex by Y. Benoist in [Ben03]. Let us stress out that the Hilbert metric on such an Ω is the hyperbolic one if and only if Ω has a $C^{1,1}$ boundary, and that its volume entropy is positive since hyperbolicity implies the non-vanishing of the Cheeger constant (see Theorem 1.5 in B. Colbois and C. Vernicos [CV07]). A result by M. Crampon (see [Cra09]) states that for a divisible open bounded convex set Ω in $\mathbb{R}^n$ whose boundary is C^1 we have $\mathrm{Ent}(\Omega) \leqslant n-1$ with equality if and only if Ω is an ellipsoid.

In the present paper we link the volume entropy to another invariant associated with a convex bodie, called the *approximability*. This name was introduced by Schneider and Wieacker in [SW81]. The approximability measures in some sense how well a convex set can be approximated by polytopes. More precisely, let $N(\varepsilon, \Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than $\varepsilon > 0$. Then the lower and upper approximability of Ω are defined by

$$\underline{a}(\Omega) := \liminf_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}, \text{ and } \quad \overline{a}(\Omega) := \limsup_{\varepsilon\to 0} \frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}. \tag{5}$$

The key inequality which is of interest in our work — obtained by Fejes-Toth [FT48] in dimension 2 and by Bronshteyn-Ivanov [BI76] in the general case — asserts that for any bounded convex set in $\mathbb{R}^n$ we have $\overline{a}(\Omega) \leqslant (n-1)/2$.

Our main result it the following

Theorem 1 (Main theorem)**.** *Given an open bounded convex set Ω in $\mathbb{R}^n$, we have*

$$2\underline{a}(\Omega) \leqslant \underline{\mathrm{Ent}}(\Omega), \text{ and } \; 2\overline{a}(\Omega) \leqslant \overline{\mathrm{Ent}}(\Omega) \tag{6}$$

with equality for $n = 2$ or $n = 3$.

The first important corollary is that it gives a proof of the entropy upper bound conjecture in dimension two and three.

Corollary 2 (Volume entropy upper bound conjecture)**.** *For any open bounded convex set Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have $\overline{\mathrm{Ent}}(\Omega) \leqslant n-1$.*

The equality case in this main theorem heavily relies on the study of polytopal Hilbert geometries. As it happens we get an optimal control of the volume of metric balls in dimension two and three for in those two cases the number of edges of a polytope is bounded from above by the number of its verticies up to a multiplicative and an additive constant. This does not hold in higher dimension, following McMullen's upper bound theorem [McM71, MS71].

The second important results concerns the two-dimensional case where we can prove that there are Hilbert geometries with intermediate volume growth.

Theorem 3 (Intermediate volume growth)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function that satisfies*

$$\liminf_{r\to+\infty} \frac{e^r}{f(r)} > 0.$$

Then there exist an open bounded convex set Ω in $\mathbb{R}^2$ and a point o in Ω, such that we have

$$\liminf_{r\to+\infty} \frac{\mathrm{Vol}_\Omega(B_\Omega(o,r))}{f(r)} > 0, \text{ and } \liminf_{r\to+\infty} \frac{\mathrm{Vol}_\Omega(B_\Omega(o,r))}{f(r)r^2} < +\infty, \tag{7}$$

and

$$\begin{aligned} \underline{\mathrm{Ent}}(\Omega) &= \liminf_{r\to+\infty} \frac{\ln f(r)}{r}, \\ \overline{\mathrm{Ent}}(\Omega) &= \limsup_{r\to+\infty} \frac{\ln f(r)}{r}. \end{aligned} \tag{8}$$

In particular there are open bounded convex sets $\Omega \subset \mathbb{R}^2$ with

- *maximal volume entropy and zero centro-projective area,*
- *zero volume entropy which are not polytopes.*

This theorem is a consequence of our method for proving the equality in dimension two in the Main theorem (see section 3) and Schneider and Wieacker [SW81] results on the approximability in dimension two. The last statement follows from our work [Ver09], where we showed that polytopes have polynomial growth of order r^2 in dimension two.

The intermediate volume growth theorem allows us to settle in a quite definite way the question of whether the entropy is a limit or not.

Corollary 4. *The volume entropy is not a limit in general. More precisely, for any $0 \leqslant \alpha \leqslant \beta \leqslant 1$ there exist an open bounded convex set Ω in $\mathbb{R}^2$ such that we have*

$$\underline{\mathrm{Ent}}(\Omega) = \alpha, \qquad \overline{\mathrm{Ent}}(\Omega) = \beta.$$

The equalities and inequalities also imply the following four new results,

Corollary 5. *Given an open bounded convex set Ω in $\mathbb{R}^n$, we have*

- $d_H \leqslant \underline{\mathrm{Ent}}(\Omega)$*, where d_H is the Hausdorff dimension of the farthest points of Ω.*
- *if $n = 2$ or 3 then $\overline{a}(\Omega)$ is a projective invariant of Ω and $\overline{\mathrm{Ent}}(\Omega) = \overline{\mathrm{Ent}}(\Omega^*)$, where Ω^* is the polar dual of Ω*
- *if $n = 2$, then $\overline{a}(\Omega) \leqslant \frac{1}{3-d}$.*

Section 2 presents the various lemmas and notions needed in section 3 to prove the main theorem, and in section 4 we present the proof of the intermediate volume growth theorem.

1. Notations and definitions

A *proper* open set in $\mathbb{R}^n$ is a set that does not contain a whole line. A non-empty proper open convex set in $\mathbb{R}^n$ will be called a *proper convex domain*. The closure of a bounded convex domain is usually called a *convex body*.

A Hilbert geometry (Ω, d_Ω) is a proper convex domain Ω in $\mathbb{R}^n$ endowed with its Hilbert distance d_Ω defined as follows: for any distinct points p and q in Ω, the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b, such that a, p, q, b appear in that order on the line. We denote by $[a, p, q, b]$ the cross ratio of (a, p, q, b), i.e.

$$[a, p, q, b] = \frac{qa}{pa} \times \frac{pb}{qb} > 1,$$

where for any two points x, y in $\mathbb{R}^n$, xy is their distance with respect to the standard Euclidean norm $||\cdot||$. Should a or b be at infinity, the corresponding ratio will be considered equal to 1. Then we define

$$d_\Omega(p, q) = \frac{1}{2}\ln[a, p, q, b].$$

Note that the invariance of the cross ratio by a projective map implies the invariance of d_Ω by such a map.

The proper convex domain Ω is also naturally endowed with the C^0 Finsler metric F_Ω defined as follows: given $p \in \Omega$ and $v \in T_p\Omega = \mathbb{R}^n$ with $v \neq 0$, the straight line passing through p with direction vector v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^- such that $p_\Omega^+ - p_\Omega^-$ and v have the same direction. Then let t^+ and t^- be the two positive numbers such that $p+t^+v = p_\Omega^+$ and $p-t^-v = p_\Omega^-$ (in other words these numbers corresponds to the amounts of time needed to reach the boundary of Ω when starting at p with the velocities v and $-v$, respectively). Then we define

$$F_\Omega(p, v) = \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad \text{and} \quad F_\Omega(p, 0) = 0.$$

Should p_Ω^+ or p_Ω^- be at infinity, then the corresponding ratio will be taken equal to 0.

The Hilbert distance d_Ω is the length distance associated to F_Ω. We shall denote by $B_\Omega(p,r)$ the metric ball of radius r centred at the point $p \in \Omega$ and by $S_\Omega(p,r)$ the corresponding metric sphere.

Thanks to that Finsler metric, we can make use of two important Borel measures on Ω.

The first one, which coincides with the Hausdorff measure associated to the metric space (Ω, d_Ω), (see exemple 5.5.13 in [BBI01]), is the *Busemann* volume that we will be denote by Vol_Ω and is defined as follows. Given any point p in Ω, let $\beta_\Omega(p) = \{v \in \mathbb{R}^n \mid F_\Omega(p,v) < 1\}$ be the open unit ball in $T_p\Omega = \mathbb{R}^n$ with respect to the norm $F_\Omega(p,\cdot)$ and let ω_n be the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^n$. Then given any Borel set A in Ω, its Busemann volume Vol_Ω is defined by

$$\mathrm{Vol}_\Omega(A) = \int_A \frac{\omega_n}{\lambda\big(\beta_\Omega(p)\big)} \mathrm{d}\lambda(p),$$

where λ denotes the standard Lebesgue measure on $\mathbb{R}^n$.

The second one, is the *Holmes-Thompson* volume on Ω that we will denote by $\mu_{HT,\Omega}$. Given any Borel set A in Ω its Holmes-Thompson volume is defined by

$$\mu_{HT,\Omega}(A) = \int_A \frac{\lambda\big(\beta_\Omega^*(p)\big)}{\omega_n} \mathrm{d}\lambda(p),$$

where $\beta_\Omega^*(p)$ is the polarvdual of $\beta_\Omega(p)$.

We can actually consider a whole family of measures as follows. Let $\mathcal{E}_n$ be the set of pointed proper open convex sets in $\mathbb{R}^n$. These are the pairs (ω, x) such that ω is a proper open convex set and x is a point in ω. We shall say that a function $f\colon \mathcal{E}_n \to \mathbb{R}$ is a *proper density* if it is positive and satisfies the three following properties

Continuity with respect to the Hausdorff pointed topology on $\mathcal{E}_n$;

Monotone decreasing with respect to the inclusion, i.e., if $x \in \omega \subset \Omega$ then $f(\Omega, x) \leqslant f(\omega, x)$.

Chain rule compatibility: for any projective transformation T one has

$$f\big(T(\omega), T(x)\big)\mathrm{Jac}(T) = f(\omega, x).$$

We will call f a *normalised proper density* if in addition f coincides with the standard Riemannian density function with respect to λ in case ω is an ellipsoid. Let us denote by PD_n the set of proper densities over $\mathcal{E}_n$.

A result of Benzecri [Ben60] states that the action of the group of projective transformations on $\mathcal{E}_n$ is co-compact. Therefore, for any pair f, g in PD_n, there exists a constant $C > 0$ ($C \geqslant 1$ for the normalised ones) such that for any $(\omega, x) \in \mathcal{E}$ one has

$$\frac{1}{C} \leqslant \frac{f(\omega, x)}{g(\omega, x)} \leqslant C. \tag{9}$$

Given a density f in PD_n and a proper open bounded convex set Ω we can define the Borel measure $\mu_{f,\Omega}$ as follows:

$$\mu_{f,\Omega}(A) = \int_A f(\Omega, p)\mathrm{d}\lambda(p),$$

where A is a borel set in Ω.

Integrating Equation (9) we obtain that for any pair f, g in PD_n, there exists a constant $C > 0$ such that for any Borel set $A \subset \Omega$ we have

$$\frac{1}{C}\mu_{g,\Omega}(A) \leqslant \mu_{f,\Omega}(A) \leqslant C\mu_{g,\Omega}(A). \tag{10}$$

We shall call *proper measures with density* the family of measures obtained in this way.

To a proper density $g \in PD_{n-1}$ we can also associate a $(n-1)$-dimensional measure, denoted by $\mu_{\cdot,g,\Omega}$, on hypersurfaces in Ω as follows. Let Σ be smooth a hypersurface, and consider for a point p in the hypersurface Σ its tangent hyperplane $H(p)$, then the measure will be given by

$$\frac{d\mu_{\Sigma,g,\Omega}}{d\sigma}(p) = \frac{d\mu_{g,\Omega\cap H(p)}}{d\sigma}(p). \tag{11}$$

Where σ denotes the Hausdorff $n-1$-dimensional measure associated with the standard Euclidean distance. In section 3 we will simply denote respectively by $\mathrm{Vol}_{n-1,\Omega}$ and Area_Ω the $(n-1)$-dimensional measure associated respectively with the Holmes-Thompson and the Busemann measures.

Let now $\mu_{f,\Omega}$ be a proper measure with density over Ω, then the volume entropies of Ω is defined by

$$\underline{\mathrm{Ent}}(\Omega) = \liminf_{r\to+\infty} \frac{\ln \mu_{f,\Omega}\big(B_\Omega(p,r)\big)}{r}, \text{ and } \overline{\mathrm{Ent}}(\Omega) = \limsup_{r\to+\infty} \frac{\ln \mu_{f,\Omega}\big(B_\Omega(p,r)\big)}{r}. \tag{12}$$

These number do not depend on either f nor p, and are equal to the spherical entropies (see Theorem 2.14 [BBV10]):

$$\underline{\mathrm{Ent}}(\Omega) = \liminf_{r\to+\infty} \frac{\ln \mathrm{Area}_\Omega\big(S_\Omega(p,r)\big)}{r}, \text{ and } \overline{\mathrm{Ent}}(\Omega) = \limsup_{r\to+\infty} \frac{\ln \mathrm{Area}_\Omega\big(S_\Omega(p,r)\big)}{r}. \tag{13}$$

2. Preliminaries on Hilbert geometries and convex bodies

2.1 Properties of the Holmes-Thompson and the Busemann measures

We recall some properties of the Holmes-Thompson and the Busemann measure.

Lemma 6 (Monotonicity of the Holmes-Thompson measure)**.** *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^n$. The Holmes-Thompson area measure is monotonic on the set of convex bodies in Ω, that is, for any K_1 and K_2 pair of convex bodies in Ω, such that $K_1 \subset K_2$ on has*

$$\mathrm{Vol}_{n-1,\Omega}(\partial K_1) \leqslant \mathrm{Vol}_{n-1,\Omega}(\partial K_2). \tag{14}$$

Proof. If $\partial\Omega$ is C^2 with everywhere positive Gaussian curvature then the tangent unit spheres of the Finsler metric are quadratically convex.

According to Álvarez Paiva and Fernandes [AF98, theorem 1.1 and remark 2] there exists a Crofton formula for the Holmes-Thompson area, from which the inequality (14) follows.

Such smooth convex bodies are dense in the set of all convex bodies for the Hausdorff topology. By approximation, it follows that inequality (14) is valid for any Ω. □

Lemma 6 associated with the Blaschke-Santalo inequality and the inequality (10) immediately implies the following result (see also [BBV10, Lemma 2.12]).

Lemma 7 (Rough monotonicity of the Busemann measure)**.** *Let (Ω, d_Ω) be a Hilbert geometry, and let p be a point in Ω. There exists a monotonic function f_Ω and a constant $C_n < 1$*

such that for all $r > 0$

$$C_n f_\Omega(r) \leqslant \mathrm{Area}_\Omega(S_\Omega(p,r)) \leqslant f_\Omega(r). \tag{15}$$

$f_\Omega(r)$ is the Holmes-Thompson area of the sphere $S_\Omega(p,r)$

Let us finish by recalling one last statement also proved in [BBV10, Lemma 2.13].

Lemma 8 (Co-area inequalities)**.** *For all $r > 0$*

$$\frac{1}{2}\frac{\omega_n}{\omega_{n-1}}\mathrm{Area}_\Omega(S_\Omega(p,r)) \leqslant \frac{\partial}{\partial r}\mathrm{Vol}_\Omega(B_\Omega(p,r)) \leqslant \frac{n}{2}\frac{\omega_n}{\omega_{n-1}}\mathrm{Area}_\Omega(S_\Omega(p,r)).$$

2.2 Upper bound on the area of triangles

In this section we bound from above independently of the two-dimensional Hilbert geometries the area of affine triangles which are subset of a metric ball, when one the vertexes is the centre of that ball. We also give a lower bound on the length of some metric segments, when their vertexes go to the boundary of the Hilbert geometry.

Lemma 9. *Let (Ω, d_Ω) be a two-dimensional Hilbert geometry. Then there exists a constant C independant of Ω, such that, for any point o in Ω and any pair of points p_ρ and q_ρ in the metric ball $B_\Omega(o,\rho)$, the area of the affine triangle $(op_\rho q_\rho)$ is less than $C\rho^2$.*

Proof. Given p_ρ and q_ρ in $B_\Omega(o,\rho)$, let p and q be the intersections of the boundary $\partial\Omega$ with the half lines $[o,p_\rho)$ and $[o,q_\rho)$ respectively. Let p' and q' be, respectively, the intersections of the halflines $[p_\rho,o)$ and $[q_\rho,o)$ with the boundary $\partial\Omega$.

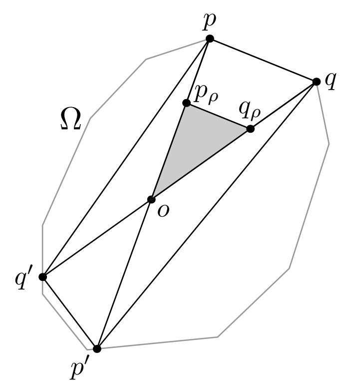

Figure 1. The area of the triangle (op_ρ, q_ρ) is bounded by $C\rho^2$.

Then the volume of the triangle $(op_\rho q_\rho)$ with respect to the Hilbert geometry of Ω is less than or equal to its volume with respect to the Hilbert geometry of the quadrilateral $(pqp'q')$. However, the distances of p_ρ and q_ρ from o remain the same in both Hilbert geometries.

Up to a change of chart, we can suppose that this quadrilateral is actually a square. This allows us to use Theorem 1 from [Ver11] which states that the Hilbert geometry of the square is bi-lipschitz to the product of the Hilbert geometries of its sides, using the identity as a map. In other words it is bi-lipschitz to the Euclidean plane, with a lipschitz constant equal to $C_0 > 1$, independent of our initial conditions.

Therefore our affine triangle is inside a Euclidean disc of radius $C_0\rho$, which implies that its area with respect to the Hilbert geometry of Ω is less than $C_0^4 \times \pi \times \rho^2$. □

Consider four points a, b, c and d in the Euclidean plane $(\mathbb{R}^2, \langle\cdot\rangle)$ such that $\mathcal{Q} = (abcd)$ is a convex quadrilateral. We assume that the scalar products $\langle\vec{ab}, \vec{bc}\rangle$ and $\langle\vec{bc}, \vec{cd}\rangle$ are positive and we let q be the intersection point between the straight lines (ab) and (cd).

Suppose that Ω is a convex domain such that the segments $[a, b]$, $[b, c]$ and $[c, d]$ belong to its boundary.

Given p a point in the convex domain Ω we denote by p' the intersection between the straight line (pq) and the segment $[b, c]$, and we define $s = bp'/bc$.

We then denote by $[b(r), c(r)]$ the image of the segment $[b, c]$ under the dilation centred at p with ratio $0 < \tanh(r) < 1$. The image of the segment $[b, c]$ under the dilation centred at q sending p' on p will be denoted by $[B, C]$.

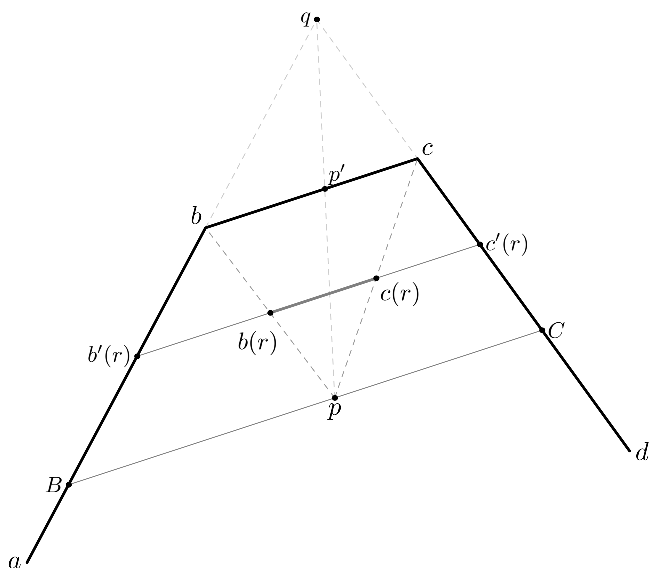

FIGURE 2. Distance estimate of Claim 10

Claim 10. *The following inequality is satified under the above assumption:*

$$d_\Omega\big(b(r), c(r)\big) \geqslant \frac{1}{2}\ln\left(\frac{bc}{s\cdot BC}\frac{\tanh(r)}{1-\tanh(r)} + 1\right) + \frac{1}{2}\ln\left(\frac{bc}{(1-s)\cdot BC}\frac{\tanh(r)}{1-\tanh(r)} + 1\right). \tag{16}$$

Proof. Straightforward computation, using the fact that the convex domain Ω is inside the convex Q obtained as the intersection of the half planes defined by the lines (ab), (bc) and (cd), and therefore

$$d_\Omega\big(b(r0, c(r)\big) \geqslant d_Q\big(b(r), c(r)\big).$$

Let $b'(r)$ be the intersection of the lines (ab) and $\big(b(r)c(r)\big)$, and let $c'(r)$ be the intersection of the lines (cd) and $\big(b(r)c(r)\big)$. Then we have

$$d_Q\big(b(r0, c(r)\big) = \frac{1}{2}\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)}\cdot\frac{c(r)b'(r)}{b(r)b'(r)}\right).$$

Let us focus on the first ratio. On the one hand $b(r)c'(r) = b(r)c(r) + c(r)c'(r)$, and on the second hand following Thales's theorem

$$\begin{aligned} b(r)c(r) &= \tanh(r)bc \\ c(r)c'(r) &= (1-\tanh(r))pC. \end{aligned} \tag{17}$$

But $pC = BC \cdot (p'c/bc) = (1-s)BC$, and therefore we obtain

$$\ln\left(\frac{b(r)c'(r)}{c(r)c'(r)}\right) = \ln\left(\frac{bc}{(1-s)\cdot BC}\frac{\tanh(r)}{1-\tanh(r)} + 1\right).$$

The second ratio is treated in the same way. □

2.3 Intrinsic and extrinsic Hausdorff topologies of Hilbert Geometries

We describe the link between the Hausdorff topology induced by an Euclidean metric with the Hausdorff topology induced by the Hilbert metric on compact subset of an open convex set.

We recall that the Lowner ellipsoid of a compact set, is the ellipsoid with least volume containing that set. In this section we will suppose, without loss of generality, that Ω is a bounded open convex set, whose Lowner ellipsoid $\mathcal{E}$ is the Euclidean unit ball and o is the center of that ball. It is a standard result that $(1/n)\mathcal{E}$ is then contained in Ω, i.e., we have the following sequence of inclusions

$$\frac{1}{n}\mathcal{E} \subset \Omega \subset \mathcal{E} \tag{18}$$

We call *asymptotic ball* of radius R centred at o the image of Ω by the dilation of ratio $\tanh R$ centred at o, and we denote it by $AsB(o,R)$.

Let us denote by *Hausdorff-Euclidean* distance the usual Hausdorff distance between compact subset of the n-dimensional Euclidean space, and by *Hausdorff-Hilbert* the metric between compact subsets of Ω, defined in the same way as the usual Hausdorff metric, but by replacing the Euclidean distance by the Hilbert distance in the definition.

We would like to relate the Hausdorff-Hilbert neighborhoods of the asymptotic ball $AsB(o,R)$ with its Hausdorff-Euclidean neighborhoods.

Proposition 11. *Let Ω be a convex domain and let o be the centre of its Lowner ellipsoid, which is supposed to be the unit Euclidean ball.*

(i) The $(1-\tanh(R))/2n$-Hausdorff-Euclidean neighborhood of the asymptotic ball $AsB(o,R)$ is contained in its $\big((\ln 3)/2\big)$-Hausdorff-Hilbert neighborhood.

(ii) For any $K>0$, the K-Hausdorff-Hilbert neighborhood of the asymptotic ball $AsB(o,R)$ is contained in its $\big(1-\tanh(R)\big)$-Hausdorff-Euclidean neighborhood.

Proof. For any point $p \in \partial\Omega$ on the boundary of Ω, and for $0<t<1$ let $\varphi_t(p) = o + t\cdot\overrightarrow{op}$. This map sends $\partial\Omega$ bijectively on the boundary of $AsB(o, \operatorname{arctanh} t)$, which we shall denote by $AsS(0, \operatorname{arctanh} t)$ and call the *asymptotic sphere* centred at o with radius $\operatorname{arctanh} t$.

First claim:

Any point of a compact set in the $(1-\tanh(R))/2n$-Hausdorff-Euclidean neighborhood of $AsB(o,R)$, either lies inside $AsB(o,R)$, or is contained in an Euclidean ball of radius $(1-\tanh(R))/2n$ centred on a point of $AsB(o,R)$.

We recall that the ball of radius $1/n$ is a subset of Ω, and thus so is the ball of radius $1/2n$, that is

$$\frac{1}{2n}\mathcal{E} \subset \frac{1}{n}\mathcal{E} \subset \Omega.$$

Let $p \in \partial\Omega$ be a point on the boundary. By convexity, the interior of $K(p)$ the convex hull of p and $1/n\mathcal{E}$ is a subset of Ω — it is the projection of a cone of basis $1/n\mathcal{E}$. Hence $\mathcal{E}_{p,\alpha}$, the image of $1/n\mathcal{E}$ by the dilation of ratio $0 < \alpha < 1$ centred at p, lies in the "cone" $K(p)$. The set $\mathcal{E}_{p,\alpha}$ is therefore an Euclidean ball of radius α/n centred at $\varphi_{1-\alpha}(p)$, and it is a subset of Ω.

A point in the Euclidean ball of radius $\alpha/2n$ centred at $\varphi_{1-\alpha}(p)$ is at a distance less or equal to $1/2\ln 3$ from $\varphi_{1-\alpha}(p)$ with respect to the Hilbert distance of $\mathcal{E}_{p,\alpha}$.

Now a standard comparison arguments states that for any two points x and y in $\mathcal{E}_{p,\alpha} \subset \Omega$ the following inequality occurs

$$d_\Omega(x,y) \leqslant d_{\mathcal{E}_{p,\alpha}}(x,y).$$

From this inequality it follows that any point in the Euclidean ball of radius $\alpha/2n$ centred at $\varphi_{1-\alpha}(p)$ is in the Hilbert metric ball centred at $\varphi_{1-\alpha}(p)$ of radius $1/2\ln 3$.

Now for any $1 \geqslant \alpha > 1 - \tanh R$, the Euclidean ball of radius $\alpha/2n$ contains the Euclidean ball of radius $(1 - \tanh R)/2n$.

This implies that for any point x in the asymptotic ball $AsB(o,R)$, the Euclidean ball of radius $(1 - \tanh R)/2n$ centred at x is contained in the Hilbert ball of radius $1/2\ln 3$ centred at the x, which allows us to obtain the first part of our claim.

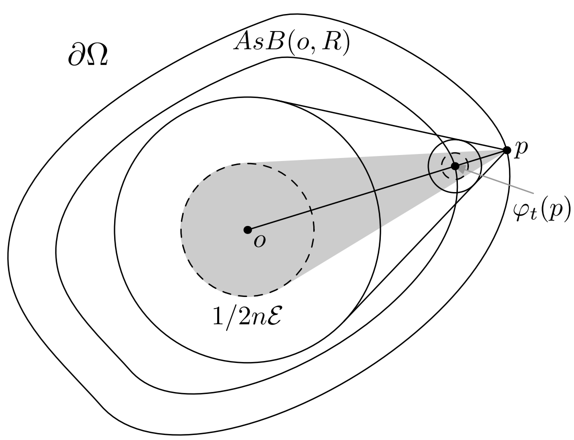

Figure 3. Illustration of Proposition 11's proof

Second claim: This follows from the fact that under our assumptions, Ω itself is in the $(1-\tanh R)$ Hausdorff-Euclidean neighborhood of the asymptotic ball $AsB(o,R)$. $\square$

Corollary 12. *Let Ω be a convex domain and let o be the centre of its Lowner ellipsoid, which is supposed to be the unit Euclidean ball.*

(i) The $(1 - \tanh(R + \ln 2))/2n$-Hausdorff-Euclidean neighborhood of $B(o,R)$ is contained in its $\ln\big(3(n+1)\big)$-Hausdorff-Hilbert neighborhood.

(ii) For any $K > 0$, the K-Hausdorff-Hilbert neighborhood of $B(o,R)$ is contained in its $\Big(1 - \tanh\big(R + K - \ln(n+1)\big)\Big)$-Hausdorff-Euclidean neighborhood.

The proof of this corrollary is a straightforward consequence of the following lemma applied to the conclusion of the Proposition 11.

Lemma 13. *Let Ω be a convex domain, and suppose that o is a point in the interior of Ω such that the unit Euclidean open ball centred at o contains Ω, and Ω contains the Euclidean closed ball centre at o of radius $1/(2n)$. Then we have*

$$\begin{aligned} B(o,R) &\subset AsB(o,R+\ln 2), \text{ and} \\ AsB(o,R) &\subset B\big(o,R+\ln(n+1)\big). \end{aligned} \tag{19}$$

This lemma is a refinement of a result of [CV04] in our case.

Proof of Lemma 13. Let x be a point on the boundary $\partial\Omega$ of Ω, and let x^* be the second intersection of the straight line (ox) with $\partial\Omega$. Then our assumption implies the next two inequalities.

$$\begin{aligned} 1/2n &< xo \leqslant 1 \\ 1/2n &< ox^* \leqslant 1 \end{aligned} \tag{20}$$

Actually the first inclusion is always true. Indeed suppose y is on the half line $[ox)$ such that $d_\Omega(o,y) \leqslant R$ which in other words implies that we have

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \leqslant e^{2R}$$

therefore

$$ox \leqslant e^{2R}\frac{ox^*}{yx^*}(ox-oy) \leqslant e^{2R}(ox-oy)$$

which implies in turn that

$$oy \leqslant \frac{e^{2R}-1}{e^{2R}}ox \leqslant \big(1-e^{-2R}\big)ox \leqslant \tanh(R+\ln 2)ox.$$

Now regarding the second inclusion: consider y a point on the half line $[ox)$ such that $oy \leqslant \tanh(R)ox$. On the one hand we have

$$\frac{ox}{yx} = \frac{ox}{ox-oy} \leqslant \frac{1}{1-\tanh(R)} = \frac{e^{2R}+1}{2}.$$

and, on the other hand thanks to the inequalities (20) we get

$$\frac{yx^*}{ox^*} \leqslant \frac{ox+ox^*}{ox^*} \leqslant 1+\frac{ox}{ox^*} \leqslant 1+2n, \tag{21}$$

which implies that

$$\frac{ox}{yx}\frac{yx^*}{ox^*} \leqslant \frac{e^{2R}+1}{2}(1+2n) \leqslant (1+2n)e^{2R} \leqslant (1+n)^2e^{2R}. \tag{22}$$

The conclusion follows. □

2.4 Polytopal dimension and approximability of convex bodies

We define the polytopal dimension of a convex body and state its properties. Noticeably we recall that an upper bound exists which is attained by the convex C^2 bodies.

Definition 14. Let Ω be an open bounded convex set in $\mathbb{R}^n$. For any $\varepsilon > 0$, let $N(\varepsilon,\Omega)$ be the smallest number of vertices of a polytope whose Hausdorff distance to Ω is less than ε.

We define the *lower and upper polytopal dimension* of Ω, respectively by

$$\underline{PD}(\Omega) := 2\liminf_{\varepsilon\to 0}\frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}, \text{ and } \overline{PD}(\Omega) := 2\limsup_{\varepsilon\to 0}\frac{\ln N(\varepsilon,\Omega)}{-\ln\varepsilon}.$$

The following result is due to R. Schneider and J. A. Wieacker [SW81]

Theorem 15 ([SW81]). *Let $\underline{a}_s := \liminf_{\varepsilon\to 0^+} N(\varepsilon,\Omega)\varepsilon^s$, then $s \to \underline{a}_s$ admits a critical value $\underline{a}(\Omega)$ called approximability number of Ω, such that, if $s > \underline{a}(\Omega)$ then $\underline{a}_s(\Omega) = 0$ and if $s < \underline{a}(\Omega)$ then $\underline{a}_s(\Omega) = \infty$.*

In the same way, we can introduce the *upper approximability number of* Ω as the critical value of $s \mapsto \overline{a}_s(\Omega)$, where

$$\overline{a}_s(\Omega) := \limsup_{\varepsilon\to 0^+} N(\varepsilon,\Omega)\varepsilon^s.$$

It is quite straightforward that the following occurs.

Proposition 16. *The polytopal dimension is twice the approximability number, i.e.,*

$$\underline{PD}(\Omega) = 2\underline{a}(\Omega), \text{ and } \overline{PD}(\Omega) = 2\overline{a}(\Omega).$$

Now the main result by E. M. Bronshteyn and L. D. Ivanov [BI76] asserts that for any convex set Ω inscribed in the unit Euclidean ball, there are no more than $c(n)\varepsilon^{(1-n)/2}$ points whose convex hull is no more than ε away from Ω in the Hausdorff topology. Which gives

Theorem 17 (E. M. Bronshteyn and L. D. Ivanov [BI76]). *Let Ω be a convex body in $\mathbb{R}^n$, then*

$$\overline{PD}(\Omega) \leqslant n-1$$

2.5 Distance function to a sphere in a Hilbert geometry

This section is an adaptation of a proof in a Minkowski space provided to the author by A. Thompson [Tom]

Let us first start by recalling the following important fact regarding the distance of a point to a geodesic in a Hilbert geometry (see Busemann [Bus55], chapter II, section 18, page 109):

Proposition 18. *Let (Ω, d_Ω) be a Hilbert Geometry. The distance function of a straight geodesic (that is given by an affine line) to a point is a peakless function, i.e., if $\gamma\colon [t_1,t_2] \to \Omega$ is a geodesic segment, then for any $x \in \Omega$ and $t_1 \leqslant s \leqslant t_2$ one has*

$$d_\Omega\big(x,\gamma(s)\big) \leqslant \max\Big\{ d_\Omega\big(x,\gamma(t_1)\big), d_\Omega\big(x,\gamma(t_2)\big)\Big\}.$$

Let us now turn our attention to metric spheres in a two dimensional Hilbert geometry.

Proposition 19. *Let (Ω, d_Ω) be a two dimensional Hilbert Geometry. Suppose o is a point of Ω, and p and q are two points on the intersection of the metric sphere $S(o,R)$ centred at o and radius R with a line passing by o. If C denotes one of the arcs of the sphere $S(o,R)$ from p to q, then for any point p' on the half line $[o,p)$, the function $\varphi(x) = d_\Omega(p',x)$ is monotonic on C.*

Proof. Let p,x,y,q be points on that order on C. We have to show that

$$d_\Omega(p',x) \leqslant d_\Omega(p',y).$$

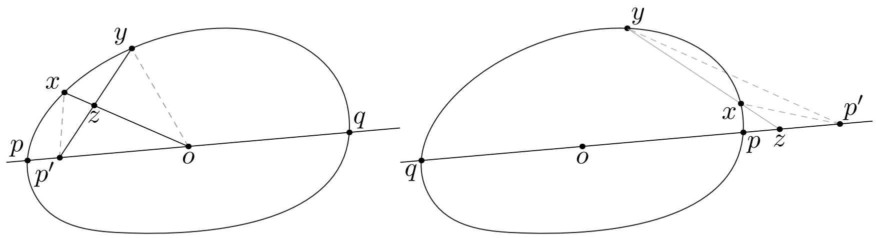

Figure 4. Monotonicity of the distance of a point to a sphere

Suppose first that that the line segments $[o,x]$ and $[p',y]$ intersects at a point z. Hence we have

$$\begin{aligned} d_\Omega(o,x)+d_\Omega(p',y) &= \big(d_\Omega(o,z)+d_\Omega(z,x)\big)+\big(d_\Omega(p',z)+d_\Omega(z,y)\big)\\ &= \big(d_\Omega(p',z)+d_\Omega(z,x)\big)+\big(d_\Omega(o,z)+d_\Omega(z,y)\big)\\ &\geqslant d_\Omega(p',x)+d_\Omega(o,y). \end{aligned}$$

now, as $d_\Omega(o,y)=d_\Omega(o,x)=R$, the result follows.

Suppose now that $[o,x]$ and $[p',y]$ do not intersect, which implies that p' is outside the ball $B(o,R)$. Then the line (yx) intersects (op) at z. Because x and y lie on the sphere of radius R, $d_\Omega(o,z)>R$. Also, as p is one of the nearest points to p' on C, we have $d_\Omega(p',z)\leqslant d_\Omega(p',p)\leqslant d_\Omega(p',y)$ Hence if apply the proposition 18 to the segment $[z,y]$ and p', as $x\in[z,y]$ we get

$$d_\Omega(p',x)\leqslant \max\big\{d_\Omega(p',z),d_\Omega(p',y)\big\}=d_\Omega(p',y).$$

□

3. Volume entropy and polytopal dimension

This section is devoted to the proof of the main theorem. This is done in two steps. The first step consists in bounding the entropy from above in dimension 2 and 3 by the approximability thanks to the study of the volume growth in polytopes. The second step is to bound from below the entropy. This is done by exhibiting a separated subset of the Hilbert geometry whose growth is bigger than the approximability. We conclude this section with the various corollaries implied.

Theorem 20. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The polytopal dimensions of Ω are bigger than the volume entropies, i.e.,*

$$\underline{\mathrm{Ent}}(\Omega)\leqslant \underline{PD}(\Omega),\ \textit{and}\ \overline{\mathrm{Ent}}(\Omega)\leqslant \overline{PD}(\Omega).$$

The proof of this theorem relies on the following stronger statement.

Theorem 21. *Let us choose a family of proper measures with density denoted by Vol_*. Then for any $n=2$ or 3 there are affine maps a_n,b_n from $\mathbb{R}\to\mathbb{R}$ and polynomials p_n,q_{n-1} of degree n and $n-1$ such that for any open convex polytope $\mathcal{P}_N$ with N vertices inside the unit Euclidean ball of $\mathbb{R}^n$ containing the ball of radius $1/2n$, one has*

$$\begin{aligned} \mathrm{Vol}_{\mathcal{P}_N}B_{\mathcal{P}_N}(o,R) &\leqslant a_n(N)p_n(R)\\ \mathrm{Vol}_{n-1,\mathcal{P}_N}S_{\mathcal{P}_N}(o,R) &\leqslant b_n(N)q_{n-1}(R). \end{aligned} \tag{23}$$

The same result holds for the asymptotic balls.

Let us stress out that our method also yields a control in terms of the vertices in higher dimension as well, using the so called upper bound conjecture proved by McMullen [McM71, MS71], but alas a polynome of degree strictly bigger than 1 replaces the affine functions. This is why we can't state the equality in the Main Theorem in higher dimensions.

Proof of theorem 21. We will have to deal with the two cases, that is dimension two and dimension three, separately.

We will start the proof considering the Holmes-Thompson measure and the asymptotic balls but we will finish with Busemann volume of the metric balls. A change of family of measures will only change the constants. The passage from the inequalities for the metric balls to the asymptotic balls is done thanks to the inclusions (19).

Let P_R be the asymptotic ball of radius R centred at o, and let us denote by $c_n = \ln(n+1)$.

Two dimensional case: The length of each edge of P_R in $\mathcal{P}_N$ is less than $2 \cdot (R + c_2)$ thanks to the triangular inequality and following the inclusions 19 in the proof of corollary 12, P_R is inside the Hilbert ball of radius $R + c_2$ centred at o of $\mathcal{P}_R$. Therefore the length of the boundary of P_R is less than $N \cdot 2 \cdot (R + c_2)$.

Following the inclusions (19) of Lemma 13 and the monotonicity of the Holmes-Thompson length (see Lemma 6) we have for all $r > 0$,

$$\text{length}_{\mathcal{P}_N}(S_{\mathcal{P}_N}(o,r)) \leqslant \text{length}_{\mathcal{P}_N}(\partial P_{r+\ln(2)}) \leqslant N \cdot 2(r + \ln 2 + c_2). \tag{24}$$

Now using the co-area inequality of Lemma 8, taking into account that the Busemann length is equal to the Holmes-Thompson length, and integrating the equation (24) over the interval $[0, R]$, we finally obtain the following inequality for the ball of radius $R > 0$

$$\text{Vol}_{\mathcal{P}_N}(B_{\mathcal{P}_N}(o,R)) \leqslant \frac{\pi}{4} \cdot N \cdot (R^2 + 2(\ln 2 + c_2)R). \tag{25}$$

The inequalities (24) and (25) are the desired results in dimension two.

Three dimensional case: Consider one of the faces of P_R, then by minimality of the Holmes-Thompson volume, the area of that face is less than the sum of the areas of the triangles obtained as the convex hull of o and an edge of the given face of P_R.

Lemma 9 applied to the polygone obtained as the intersection of the affine space containing such a triangle and $\mathcal{P}_N$ implies that the area of such a triangle is less than $C(R+c_3)^2$, for $C > 1$ some constant independent of R.

Therefore, if $e(N)$ is the number of edges of $\mathcal{P}_N$, the area of asymptotic sphere ∂P_R is less than $e(N)C(R + c_3)^2$.

Let $f(N)$ be the number of faces of $\mathcal{P}_N$ and let us recall Euler's formula:

$$N - e(N) + f(N) = 2.$$

Each face being surrounded by at least three edges and each edge belonging to two faces, one has the classical inequality (where equality is obtained in a simplex),

$$3f(N) \leqslant 2e(N).$$

Combining the previous two inequalities we get a linear upper bound of the number of edges by the number of vertexes as follows:

$$2 \leqslant N - (1/3)e(N) \Rightarrow e(N) \leqslant 3N - 6.$$

Hence the area of ∂P_R is less than $(3N-6)\cdot C \cdot (R+c_3)^2$.

Following the inclusions (19) of Lemma 13 and the monotonicity of the Holmes-Thompson length (see Lemma 6) we have for all $r > 0$

$$\mathrm{Vol}_{2,\mathcal{P}_N}(S_{\mathcal{P}_N}(o,r)) \leqslant \mathrm{Vol}_{2,\mathcal{P}_N}(\partial P_{r+\ln 2}) \leqslant (3N-6)C(r+\ln 2+c_3)^2. \tag{26}$$

the rough monotonicity of the Busemann measure (see Lemma 7) implies then that for all $r > 0$ one has

$$\mathrm{Area}_{\mathcal{P}_N}(S_{\mathcal{P}_N}(o,r)) \leqslant (3N-6)\cdot C \cdot (r+\ln 2+c_3)^2. \tag{27}$$

Let us now take into account the co-area inequality (see Lemma 8) to obtain that

$$\frac{\partial}{\partial r}\mathrm{Vol}_{\mathcal{P}_N}(B_{\mathcal{P}_N}(o,r)) \leqslant 2\cdot(3N-6)\cdot C\cdot(r+\ln 2+c_3)^2. \tag{28}$$

From which we finally deduce by integrating over the interval $[0,R]$ that

$$\mathrm{Vol}_{\mathcal{P}_N}(B_{\mathcal{P}_N}(o,R)) \leqslant 2\cdot(N-2)\cdot C\cdot\big((r+\ln 2+c_3)^3 - c_3^3\big). \tag{29}$$

which concludes our proof in the three dimensional case. □

Le us remark that if we link this to our study of the asymptotic volume of the Hilbert geometry of polytopes [Ver13] we obtain the following corollary

Corollary 22. *Let $\mathcal{P}_N$ be an open convex polytope with N vertices in $\mathbb{R}^n$, for $n = 2$ or 3, then there are three constants α_n, β_n and γ_n such that for any point $p \in \mathcal{P}_N$ one has*

$$\alpha_n \times N \leqslant \liminf_{R\to+\infty} \frac{\mathrm{Vol}_{\mathcal{P}_N} B_{\mathcal{P}_N}(p,R)}{R^n} \leqslant \beta_n \times N + \gamma_n$$

Now let us come back to our initial problem and see how theorem 21 implies theorem 20.

Proof of theorem 20 . We remind the reader that $\mathrm{Vol}_{n-1,\Omega}$ stands for the $n-1$-dimensional Holmes-Thompson measure. Let o be the centre of the Lowner ellipsoid of Ω which is supposed to be the unit Euclidean ball. We consider R large enough such that the Euclidean ball of radius $1/2n$ is inside all the convex sets involved in the sequel.

The idea of the proof is to replace, for each real positive number R large enough, the convex set Ω by a convex polytope $\mathcal{P}_R \subset \Omega$ such that its asymptotic ball P_R is in the $1-\tanh(R)/2n$-Euclidean neighborhood of the asymptotic ball $AsB_\Omega(o,R)$. Let us insist on the fact that the convex polytope $\mathcal{P}_R$ will depend on R. The choice done would guaranty that the exponential volume growth, with respect to the geometry of Ω, of the two families of asymptotic balls, that is $(P_R)_{R\in\mathbb{R}}$ and $(AsB_\Omega(o,R))_{R\in\mathbb{R}}$ are the same.

Then using Theorem 21 we will bound from above the area in dimension three or the perimeter in dimension two of the convex polytope P_R by a function depending linearly on the number of verticies of P_R and polynomialy on R. This will allow us to conclude.

Given R, among all polytopes in the asymptotic ball $AsB_\Omega(o,R)$ and at Euclidean-Hausdorff distance less than $\big(1-\tanh(R)\big)/2n$ from it, consider a polytope P_R with the minimal number of verticies $N(R) := N\big(\Omega, \frac{1-\tanh(R)}{2n\tanh(R)}\big)$.

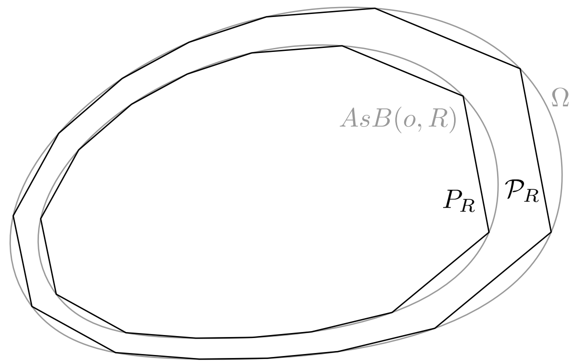

Figure 5. The asymptotic ball and an approximating polytope

Proposition 11 implies that the convex polytope P_R lies in the $(\ln 3)/2$-Hilbert-Hausdorff neighborhood of the asymptotic ball $AsB_\Omega(o,R)$, and contains the asymptotic ball of radius $R-C$, for C a constant independent of R (*e.g.* one can take $C=\ln(n+1)+\ln 6$ following (i) and (ii) of Corollary 12 and Lemma 13).

Thanks to the monotonicity of the Holmes-Thompson measure (see Lemma 6) we know that the area of P_R is less than the area of the asymptotic ball $AsB_\Omega(o,R)$, but bigger than the area of the asymptotic ball of radius $R-C$.

Therefore one has

$$\mathrm{Vol}_{n-1,\Omega}(AsB_\Omega(o,R-C)) \leqslant \mathrm{Vol}_{n-1,\Omega}(P_R) \leqslant \mathrm{Vol}_{n-1,\Omega}(AsB_\Omega(o,R)) \tag{30}$$

From the equation (30) we deduce that the logarithms of the areas of P_R and $AsB_\Omega(o,R)$ are asymptotically the same in the following sense

$$\lim_{R\to+\infty} \frac{\ln \mathrm{Vol}_{n-1,\Omega}(AsB_\Omega(o,R))}{\ln \mathrm{Vol}_{n-1,\Omega}(P_R)} = 1. \tag{31}$$

Let us denote by $\mathcal{P}_R$ the image of P_R by the dilation of ratio $1/(\tanh R)$. This is the dilation sending $AsB_\Omega(o,R)$ to Ω. Hence, by construction, $\mathcal{P}_R$ is in Ω and therefore we have

$$\mathrm{Vol}_{n-1,\Omega}(P_R) \leqslant \mathrm{Vol}_{n-1,\mathcal{P}_R}(P_R). \tag{32}$$

Now thanks to Theorem 21, for $n=2$ or $n=3$ and $R>0$ such that $\tanh(R)>3/4$, there are two constants a_n,b_n and a polynomial Q_n of degree n such that

$$\mathrm{Vol}_{n-1,\Omega}(P_R) \leqslant (a_n N(R)+b_n)Q_n(R) \tag{33}$$

Now

$$\frac{\ln(N(R))}{R} = 2\frac{\ln(N(R))}{\ln(e^{2R})} = 2\frac{\ln(N(R))}{\ln\left(\frac{1-\tanh R}{2n\tanh(R)}\right)} \frac{\ln\left(\frac{1-\tanh(R)}{2n\tanh(R)}\right)}{\ln(e^{2R})}$$

which implies that

$$\liminf_{R\to+\infty} \frac{\ln(N(R))}{R} = \underline{PD}(\Omega), \text{ and } \limsup_{R\to+\infty} \frac{\ln(N(R))}{R} = \overline{PD}(\Omega).$$

The conclusion follows from the inequality (33) and the limit (31). □

The following corollary follows from Bronshteyn and Ivanov's Theorem 17 which states that $\overline{PD(\Omega)} \leqslant n-1$.

Corollary 23. *Let Ω be an open bounded convex set in $\mathbb{R}^n$, for $n=2$ or 3, then*

$$\overline{\mathrm{Ent}}(\Omega) \leqslant n-1.$$

We are now going to study the reverse inequality.

Theorem 24. *Let Ω be an bounded convex domain in $\mathbb{R}^n$. The volume entropies of Ω are bigger or equal to the polytopal dimensions of Ω, i.e.,*

$$\underline{PD}(\Omega) \leqslant \underline{\mathrm{Ent}}(\Omega), \text{ and } \overline{PD}(\Omega) \leqslant \overline{\mathrm{Ent}}(\Omega).$$

Proof of Theorem 24. Without loss of generality we suppose that the Euclidean unit ball is the Lowner ellipsoid of Ω, and o is the centre of that ball.

The proof consists in showing that for a good positive δ and any positive real number R there exists a δ-separated set $\mathcal{S}_R$ in the metric ball of radius $B(o, R+2\delta)$, such that the convex closure P_R of that set contains the ball $B(o,R)$. In this case the δ-separated set will have more points than the number of verticies of a convex polytope in the annulus $B(o,R+2\delta)\setminus B(o,R)$ with the minimum number of verticies among polytopes in that annulus. This will then allow us to bound from below the number of points in the δ-separated by the number $N(\varepsilon,\Omega)$ of definition 14, where ε will be a function of R. To conclude we will take into account that the union of the open balls of radius $\delta/2$ centred at the point of the δ-separated set $\mathcal{S}_R$ are disjoints and are in the ball $B(o,R+3\delta)$.

Consider the $(\ln 3)/2$-Hilbert neighborhood of the metric ball $B(o,R)$, that is

$$V(R) = B(o, R + (\ln 3)/2),$$

and take a maximal $\delta = (\ln 3)/4$-separated set $\mathcal{S}_R$ on its boundary. This set contains $\#\mathcal{S}_R$ points.

Now let us take the convex hull $\mathcal{C}_R$ of these points. This is a polytope with $N_2(R) \leqslant \#\mathcal{S}_R$ vertices.

Claim 25. *The polytope $\mathcal{C}_R$ is included in the 2δ-Hilbert neighborhood of $B(o,R)$ and contains $B(o,R)$.*

Notice that if the claim holds, then for some real constant c independent of R (see corollary 12 once again), we have

$$\#\mathcal{S}_R \geqslant N_2(R) \geqslant \widetilde{N}(R-c) = N(1-\tanh(R-c)/4, AsB(o,R-c)). \tag{34}$$

proof of claim 25. First notice that $V(R)$ is a convex set (see Busemann [Bus55], chapter II, section 18, page 105). Therefore the convex hull is inside the 2δ-Hilbert neighborhood of $B(o,R)$, that is $V(R)$.

Now let us suppose by contradiction that $\mathcal{C}_R$ does not contain $B(o,R)$. Hence there exists some points q in $B(o,R)$ which is not in $\mathcal{C}_R$.

We will show that we can find a point on the sphere $S(o,R+2\delta)$ which is at a distance bigger that δ from all points of $\mathcal{S}_R$, which will contradict its maximality.

Under our assumption, the Hahn-Banach separation theorem asserts that there exists a linear form a, some constant c and a hyperplane $H=\{x \mid a(x)=c\}$ which separates q and $\mathcal{C}_R$, i.e., $a(q) > c$ and $a(x) < c$ for all $x \in \mathcal{C}_R$. Consider then $H_q = \{x \mid a(x) = a(q)\}$ the parallel hyperplane to H containing q. Let us say that a point x such that $a(x) \geqslant a(q)$ is *above* the hyperplane H_q.

Then let us define by $V_o' = \{x \in \partial V(R) \mid a(x) \geqslant a(q)\}$ the part of the boundary of $V(R)$ which is above H_q.

Now we want to metrically project each point of V_o' onto H_q, that is to say that to each point of V_o' we associate its closest point on H_q. However if Ω is not strictly convex, the projection might not be unique (see the appendix A), that is why we are going to distinguish two cases.

First case: The convex set Ω is strictly convex, then the metric projection is a map from V_o' to H_q and it is continuous, furthermore the point on $H_q \cap V_o'$ are fixed and by convexity $H_q \cap V_o'$ is homeomorphic to a $n-2$-dimensional sphere. Therefore by Borsuk-Ulam's theorem (or its version known as the *antipodal map theorem*), there is a point p on V_o' whose metric projection is q.

Now as p is on the boundary of $V(R)$, that is the sphere $B(o, R+2\delta)$, and q is in $B(o,R)$ we necessarily have

$$d_\Omega(p,q) \geqslant (\ln 3)/2.$$

hence for all points x in $H_q \cap V_o'$, we have

$$d_\Omega(p,x) \geqslant d_\Omega(p,q) \geqslant (\ln 3)/2.$$

Second case: The convex set Ω is not strictly convex. Then let us approximate it by a smooth and strictly convex set Ω' such that $\Omega \subset \Omega'$, and for all pair of points $x, y \in V(R)$,

$$\frac{2}{3} \times d_{\Omega'}(x,y) \geqslant d_\Omega(x,y) \geqslant d_{\Omega'}(x,y). \tag{35}$$

Then metrically project V_o' onto H_q with respect to Ω'. By the same argument as in the first case, we obtain a point p such that for all x in $H_q \cap V_o'$ we have

$$d_{\Omega'}(p,x) \geqslant d_{\Omega'}(p,q) \geqslant \frac{3}{2} d_\Omega(p,q) \geqslant \frac{3}{4}(\ln 3)$$

which also implies by the inequalities (35) that for all x in $H_q \cap V_o'$ we have

$$d_\Omega(p,x) \geqslant 3(\ln 3)/4.$$

In either cases, using the Lemma 19 of the section 2.5, we deduce that all points on ∂V_R at distance less or equal to $(\ln 3)/4$ from p are above H_q and are therefore contained in V_o'. We then infer that there are no points of $\mathcal{S}_R$ at distance less or equal to $(\ln 3)/4$ from p, which contradicts the maximality of the set $\mathcal{S}_R$. $\square$

Now consider the union of the balls of radius $\delta/2$ centred at the points of $\mathcal{S}_R$. This union is a subset of the ball $B(o, R+3\delta)$ and the balls are mutually disjoint. Now following our paper [Ver13], there exists a constant a_n such that for any open proper convex Ω and $x \in \Omega$, the volume of the ball of radius r centred at x is at least $a_n r^n$. Hence from this fact and the inequality (34) we get that for all $R > 0$,

$$\mathrm{Vol}_\Omega\big(B(o, R+3\delta)\big) \geqslant \#\mathcal{S}_R \cdot a_n\delta^n \geqslant N(1-\tanh(R-c)/4, AsB(o, R-c)) \cdot a_n\delta^n. \tag{36}$$

Now if we take the logarithm of the previous inequalities, divide by R and take the $\liminf$ and $\limsup$ we obtain our theorem. $\square$

A point x of a convex body K is called a *farthest point* of K if and only if, for some point $y \in \mathbb{R}^n$, x is farthest from y among the points of K. The set of farthest points of K, which are special exposed points, will be denoted by exp^*K. Thus a point $x \in K$ belongs to exp^*K if and only if there exists a ball which circumscribes K and contains x in its boundary.

we thus have the following corollary in dimension 2,

Corollary 26. *Let Ω be a plane Hilbert geometry, and let d_M be the Minkowski dimension of extremal points and d_H the Hausdorff dimension of the set $exp^*\Omega$ of farthest points then we have the following inequalities*

$$d_H \leqslant \underline{\mathrm{Ent}}(\Omega) \leqslant \overline{\mathrm{Ent}}(\Omega) \leqslant \frac{2}{3-d_M}. \tag{37}$$

The left hand side inequality remains valid for higher dimensional Hilbert geometries.

Proof. The left hand side of inequality (37) comes from R. Schneider and J. A. Wieacker [SW81], whereas the right hand one is the First main Theorem in G. Berck, A. Bernig and C. Vernicos [BBV10]. □

Remark 27. Inequality (37) induces a new result concerning the approximability in dimension 2, as it implies that

$$\overline{a}(\Omega) \leqslant \frac{1}{3-d}.$$

Lastly we are also able to prove the following result which relates the entropy of a convex set and the entropy of its polar body.

Corollary 28. *Let Ω be a Hilbert geometry of dimension 2 or 3, then*

$$\underline{\mathrm{Ent}}(\Omega) = \underline{\mathrm{Ent}}(\Omega^*), \text{ and } \quad \overline{\mathrm{Ent}}(\Omega) = \overline{\mathrm{Ent}}(\Omega^*)$$

Proof. It suffices to prove that the approximability of a convex body Ω containing the origin and its polar Ω^* are equal. Without loss of generality we can assum that the unit ball is Ω's John's ellipsoid. Hence Ω is contained in the ball of radius the dimension and its polar contains the ball of radius the inverse of the dimension and is included in the unit ball. Now, notice that for ε small enough, if P_k is a polytope with k vertexes inside the ε-Hausdorff neighborhood of Ω, then its polar P_k^* is a polytope with k faces containing Ω^* and contained in its $\varepsilon \cdot C$-Hausdorff neighborhood, for some constant C depending only on the dimension. A known fact (see Gruber [Gru07] section 11.2) states that the approximability can be computed either by minimising the vertexes or the faces. Hence $\underline{a}(\Omega) = \underline{a}(\Omega^*)$ and $\overline{a}(\Omega) = \overline{a}(\Omega^*)$. The statement therefore follows from the Main Theorem. □

4. Intermediate growth

The intermediate volume growth will follow from Theorem 21 and the following Proposition, which allows us to control both the length of sphere and their volume in dimension 2 from below, thanks to the number of verticies of an ad-hoc approximating polytope, in the fashion of Theorem 21, except that here the lower bounds depend on Ω.

Proposition 29. *Let Ω be an open bounded convex set in $\mathbb{R}^2$ whose Lowner ellipsoid is the Euclidean unit ball centered at $o \in \Omega$. Let $N(\epsilon, \Omega)$ be the minimal number of verticies of a* *polygon containing Ω at Euclidean-Hausdorff distance less that ϵ from Ω, and to any positive real number R let $N(R) := N(\frac{1-\tanh(R)}{4\tanh(R)}, \Omega)$.*

Then there exists three constants R_2, K_2 and C_2 independant of Ω, such that for all real numbers $R > R_2$ we have

$$\begin{aligned} \mathit{Length}_\Omega\big(S_\Omega(o,R)\big) &\geqslant \big(N\big(R-(3/2)\ln 3\big)-2\big)K_2, \\ \mathrm{Vol}_\Omega\big(B_\Omega(o,R+K_2/2)\big) &\geqslant \big(N\big(R-(3/2)\ln 3\big)-2\big)C_2(K_2)^2. \end{aligned} \tag{38}$$

The same result holds for the asymptotic balls with $R > R_2 + \ln 2$.

We want to stress out once again that there is actually no loss in generality in supposing the Euclidean unit ball to be the Lowner ellipsoid of Ω.

Proof. For any real positive number R let $\epsilon(R) = \big(1-\tanh(R)\big)/4$.

The idea is to built a convex polygone in the $\epsilon(R)$-neighborhood of an asymptotic ball of radius R in a way we can control uniformly from below the length of the edges.

More precisely we have the following.

Claim 30. *There exist a convex polygone $\mathcal{P}_R$ such that*

- *$\mathcal{P}_R$ contains the asymptotic ball $AsB(o,R)$ and is in its $\epsilon(R)$-Hausdorff-Euclidean neighborhood;*
- *All the edges of $\mathcal{P}_R$ but one are tangent to $AsB(o,R)$ and all its vertexes belong to the boundary $\partial_R AsB$ of the $\epsilon(R)$-Hausdorff neighborhood of the asymptotic ball $AsB_\Omega(o,R)$.*

This claim is a consequence of the following algorithm:

Step 1 Draw one tangent to $AsB_\Omega(o,R)$, it will meet the boundary $\partial_R AsB$ of its $\epsilon(R)$-Hausdorff neighborhood at two points x_1 and x_2, where $\overrightarrow{ox_1}$, $\overrightarrow{ox_2}$ are positively oriented.

Step 2 We start from x_2 and draw the second tangent to $AsB_\Omega(0,R)$ passing by x_2. This second tangent will meet the boundary $\partial_R AsB$ at a second point x_3.

Step 3 for $k > 2$, if the second tangent t_{k+1} to $AsB_\Omega(0,R)$ passing by x_k has its second intersection with $\partial_R AsB$ on the arc of from x_1 to x_k (in the orientation of the construction), we stop and consider for $\mathcal{P}_R$ the convex hull of $x_1,\dots,x_k$, otherwise we take for x_{k+1} that second intersection of the tangent t_{k+1} with $\partial_R AsB$ and start again that step.

This algorithm will necessarily finish, because by convexity the arclength of $x_i x_{i+1}$ on $\partial_R AsB$ built this way is bigger than $2\epsilon(R)$. At the end of this algorithm we obtain, by minimality, a polygon which has at least $N(R) = N\big(\epsilon(R), AsB_\Omega(o,R)\big) = N\big(\epsilon(R)/\tanh(R),\Omega\big)$ edges.

Recall that Proposition 11 guaranties us that the $\epsilon(R)$-Euclidean neighborhood of the asymptotic ball $AsB_\Omega(o,R)$ is included in its $(\ln 3)/2$-Hausdorff-Hilbert neighborhood and therefore, taking into account the inclusions (19), we obtain

$$B_\Omega(o,R-\ln 2) \subset AsB_\Omega(o,R) \subset \mathcal{P}_R \subset B_\Omega\big(o,R+(3/2)\ln 3\big).$$

Moreover, the length coincides with the Holmes-Thompson 1-dimensional measure. Therefore, the monotonicity of the later, as seen in Lemma 6, implies the following inequalities:

$$\begin{aligned} \mathrm{length}_\Omega S_\Omega(o,R-\ln 2) \leqslant& \mathrm{length}_\Omega \partial AsB_\Omega(o,R) \\ &\leqslant \mathrm{length}_\Omega \partial\mathcal{P}_R \leqslant \mathrm{length}_\Omega S_\Omega(o,R+3/2\ln 3). \end{aligned} \tag{39}$$

Now let $\mathfrak{P}_R$ be the image of $\mathcal{P}_R$ under the dilation of ratio $\tanh(R)^{-1}$ centred at o. By construction $\mathfrak{P}_R$ contains Ω, which implies

$$\mathrm{Length}_{\mathfrak{P}_R} \partial \mathcal{P}_R \leqslant \mathrm{length}_\Omega \partial \mathcal{P}_R.$$

Therefore it sufficies to prove the following claim:

Claim 31. *Let $I(R) \in \partial_R AsB$ be a vertex of $\mathcal{P}_R$, such that the two edges containing $I(R)$ are tangent to $AsB_\Omega(o,R)$ at $b(R)$ and $c(R)$. Then for any $R > \tanh^{-1}(1/2) = R_2$*

$$d_\Omega(b(R), c(R)) \geqslant d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geqslant \ln(6/5) = K_2.$$

Indeed, let us assume that claim 31 is true, and for $R > r_2$ consider a vertex v of $\mathcal{P}_R$ whose incident edges are tangent to $AsB(o,R)$. Let b and c the two points of tangency, then by the triangle inequality,

$$d_\Omega(b,v) + d_\Omega(c,v) \geqslant d_\Omega(b,c) \geqslant K_2.$$

Therefore the length of $\mathcal{P}_R$ is bigger than $\big(\widetilde{N}(R)-2\big)K_2$, where $\widetilde{N}(R)$ is number of edges of $\mathcal{P}_R$ (because of the possible exception at x_1 and the last point of the construction above). Hence taking $R_2 = r_2 + (3/2)\ln 3$, thanks to the equation (39), we get for $R > R_2$

$$\mathrm{Length}_\Omega\big(S_\Omega(o,R)\big) \geqslant \Big(\widetilde{N}\big(R-(3/2)\ln 3\big) - 2\Big)K_2, \tag{40}$$

and as $\widetilde{N}\big(R-(3/2)\ln 3\big) \geqslant N\big(R-(3/2)\ln 3\big)$ the first inequality in (38) is proved.

Now concerning the volume of the ball, Claim 31 and Proposition 19 imply that the contact points of the edges of $\mathcal{P}_R$ with $AsB_\Omega(o,R)$ form a K_2 separated set. Hence we can conclude in the same way as we did during the Proof of Theorem 24, *i.e.*, the balls of radius $K_2/2$ centred at those points are disjoint and included in the metric ball $B_\Omega\big(o, R+(3/2)\ln 3 + (K_2/2)\big)$. Now following [Ver13], there exists a constant C depending only on the dimension such that the volume of the ball of radius r is at least $C \cdot r^2$. Hence we obtain that

$$\mathrm{Vol}_\Omega B_\Omega\big(o, R+(3/2)\ln 3 + (K_2/2)\big) \geqslant (\widetilde{N}(R)-2)\cdot C \cdot (K_2/2)^2, \tag{41}$$

and the last inequality (38) follows onece again from the inequality $\widetilde{N}(R) \geqslant N(R)$.

Proof of the Claim 31.

Let $a(R)$ (resp. $d(R)$) be the opposite vertex to $I(R)$ on the edge containing $b(R)$ (resp. $c(R)$).

Now let us consider the images I, a,b, c and d of the five points $I(R)$, $a(R)$, $b(R)$, $c(R)$ and $d(R)$ by the dilation of ratio $1/\tanh R$ centred at o. Then we are in the same configuration as in the claim 10, with $\mathfrak{P}_R$ instead of Ω. Let $u(R) = \frac{bc}{BC}\frac{\tanh(R)}{1-\tanh(R)}$, then following (16) we have

$$d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geqslant \frac{1}{2}\ln\Big(1 + \frac{u(R)+u(R)^2}{s(1-s)}\Big).$$

Therefore we need to obtain a lower bound for $u(R)$. To do this, let p be the intersection of the line oI with the lines (bc). Then thanks to Thales's theorem we have

$$\frac{BC}{bc} = \frac{oI}{pI} = \frac{op+pI}{pI} = 1 + \frac{op}{pI}$$

Concerning the distance op, recall that the unit ball centred at o is the Lowner ellipsoid of Ω and therefore we get $op \leqslant \frac{1}{\tanh(R)}$, because by convexity p is in Ω.

Regarding the distance pI, as $I(R)$ is on the boundary of the $(1-\tanh(R))/4$ Euclidean neighborhood of $AsB(o,R)$, I is on the boundary of the $\big(1-\tanh(R)\big)/4\tanh(R)$ neighborhood of Ω. Hence we obtain $pI \geqslant \big(1-\tanh(R)\big)/4\tanh(R)$, because the segment $[p,I]$ intersects Ω. This way we obtain

$$\frac{BC}{bc} \leqslant 1 + \frac{4}{1-\tanh(R)}$$

which in turn implies that

$$1 \leqslant \frac{5-\tanh(R)}{1-\tanh(R)}\frac{bc}{BC} \leqslant \frac{5}{1-\tanh(R)}\frac{bc}{BC}.$$

Hence

$$\frac{\tanh(R)}{5} \leqslant u(R) \tag{42}$$

Therefore if $\tanh(R_2) = 1/2$ then for all $R > R_2$ we get $10u(R) > 1$.

Finally using the fact that $s(1-s) \leqslant 1/4$ and taking $R > R_2$ we get

$$d_{\mathfrak{P}_R}\big(b(R), c(R)\big) \geqslant \frac{1}{2}\ln\left(1 + \frac{2}{5} + \frac{1}{25}\right) = \ln(6/5) > 0.18.$$

□

Proof of the intermediate volume growth theorem. Following Schneider and Wieacker [SW81, theorem 4, p. 154] and its proof, for any increasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\liminf_{r\to+\infty} \frac{e^r}{f(r)} > 0$$

there exists a convex set Ω_f such that

$$0 < \liminf_{r\to+\infty} \frac{N(1-\tanh(r),\Omega_f)}{f(r)} \leqslant \limsup_{r\to+\infty} \frac{N(1-\tanh(r),\Omega_f)}{f(r)} < +\infty. \tag{43}$$

Let us denote $N(r) = N(1-\tanh(r),\Omega_f)$.

Now let o be the center of the lowner ellipsoid of Ω_f. Following Proposition 29 we then have for $K_2 = \ln(6/5)$ and $r > 0$ satisfying $\tanh(r - (3/2)\ln 3 - K_2/2) \geqslant 1/2$ that

$$\mathrm{Vol}_{\Omega_f}(B(o,r)) \quad \geqslant \quad \big(N\big(r-(3/2)\ln 3 - K_2/2\big) - 2\big)C(K_2)^2. \tag{44}$$

This inequality implies that

$$\liminf_{r\to+\infty} \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{f(r)} \geqslant C(K_2)^2 \liminf_{r\to+\infty} \frac{N\big(r-(3/2)\ln 3 - K_2/2\big) - 2}{f(r)}. \tag{45}$$

Now using inequalities (30) to (33) in Theorem 20 proof's we get the existence of three constants a, b and c such that if $K = \ln 18$ and $r > 0$ is a real number satisfying $\tanh(r-C) > 3/4$ then

$$\mathrm{Vol}_{\Omega_f}(AsB(o,r-C)) \leqslant N\big(\frac{(1-\tanh(r))}{4\tanh(r)},\Omega_f\big)(ar^2+br+c). \tag{46}$$

The inclusion $B(o, r-\ln(2)-C) \subset AsB(o,r-C)$ given by (19) in Lemma 12 proof's allow us to obtain the next inequality:

$$\mathrm{Vol}_{\Omega_f}(B(o,r-C-\ln 2)) \leqslant N\Big(\frac{(1-\tanh(r))}{4\tanh(r)},\Omega_f\Big)(ar^2+br+c), \tag{47}$$

which in turm implies that

$$\limsup_{r\to+\infty} \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{r^2 f(r)} \leqslant a \times \limsup_{r\to+\infty} \frac{N\left(\frac{(1-\tanh(r))}{4\tanh(r)}, \Omega_f\right)}{f(r)}. \tag{48}$$

Combining both inequalities (44) and 46) and using the asymptotic comparison(43) we finally conclude that

$$\liminf_{r\to+\infty} \frac{\ln \mathrm{Vol}_{\Omega_f}(B(o,r))}{r} = \liminf_{r\to+\infty} \frac{\ln f(r)}{r}.$$

In the above proofs we can replace lim inf by lim sup.

To obtain the penultimate statement consider $f(r) = e^r/r^3$, and apply our result to get a convex set Ω_f whose entropy is 1. However, by definition of the centro-projective area and our result in the two dimensional case [BBV10] we have

$$\begin{aligned}\mathcal{A}_o(\Omega_f) = \lim \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{\sinh r} &= \limsup \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{\sinh r} \\ &= \limsup \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{e^r r^{-1}} \times \frac{e^r}{r \sinh r} = 0. \end{aligned}\tag{49}$$

For the last statement take $f(r) = r^3$ and apply our result to get a convex Ω_f such that

$$\limsup \frac{\mathrm{Vol}_{\Omega_f}(B(o,r))}{r^2} = \limsup \frac{r\mathrm{Vol}_{\Omega_f}(B(o,r))}{r^3} = +\infty$$

hence following our paper [Ver13], Ω_f is not a polytope. Furthermore the entropy of such a Ω_f is zero as we have $\limsup_\infty \ln(r^3)/r = 0$.

□

To conclude this section let us show how Corollary 4 related to the values attained by the lower and upper volume entropies easily follows: Suppose first that $0 < \alpha \leqslant \beta \leqslant 1$, and start by considering a sequence $(U_n)_{n\in\mathbb{N}}$ defined for some $x > 0$ by $U_0 = e^{bx}$, and for all $k \geqslant 0$ by

$$U_{2k+1} = e^{\alpha U_{2k}}, \text{ and } U_{2k+2} = e^{\beta U_{2k+1}}.$$

Then take an increasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $r \in \mathbb{R}$,

$$e^{\alpha r} \leqslant f(r) \leqslant e^{\beta r},$$

and $f(U_n) = U_{n+1}$ for all $n \geqslant 0$. We can define such a function piecewise linearly.

If $\alpha = 0$, replace $r \mapsto e^{\alpha r}$ by $r \mapsto 2r$ above and take $U_{2k+1} = 2U_{2k}$ for all $k \geqslant 0$.

Appendix A. Metric projection in a Hilbert geometry

The following is a reformulation and a detailed proof of a statement found in section 21 and 28 of Busemann-Kelly's book [BK53] in any dimension.

Proposition 32. *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^n$. Let p be a point of Ω and H an hyperplane intersecting Ω. Then $q \in H \cap \Omega$ is a metric projection of p onto H, i.e.,*

$$d_\Omega(p, H) = d_\Omega(p, q),$$

if and only if $\partial\Omega$ has, at its intersection with the straight line (pq), supporting hyperplanes concurrent with H (the intersection of these three hyperplanes is an $n-2$-dimensional affine space).

Proof. Let us suppose first that such concurrent support hyperplanes exists. Let x and y be the intersections of the line (pq) with $\partial\Omega$. Assume that ξ and η are supporting hyperplanes of $\partial\Omega$ respectively at x and y whose intersection with H is the $n-2$-affine space W. Let us show that for any $p' \in (pq)$ and any $q' \in H$ we have

$$d_\Omega(p', q') \geqslant d_\Omega(p', q). \tag{50}$$

Let us suppose that x is on the half line $[qp')$ and y on the half line $[p'q)$ and denote by x' and y' the intersection of $\partial\Omega$ with the half line $[q'p')$ and $[p'q')$ respectively. Then let x_0 be the intersection of ξ with the line $(p'q')$ and y_0 the intersection of $(p'q')$ with η. By Thales' theorem, the cross-ratio of $[x_0, p', q', y_0]$ is equal to the cross ratio of $[x, p', q, y]$ and standard computation shows that $[x_0, p', q', y_0] \leqslant [x', p', q', y']$, with equality if an only if $x_0 = x'$ and $y' = y_0$. Hence the inequality (50) holds, and if the convex set is strictly convex, this inequality is always strict, for $q' \neq q$.

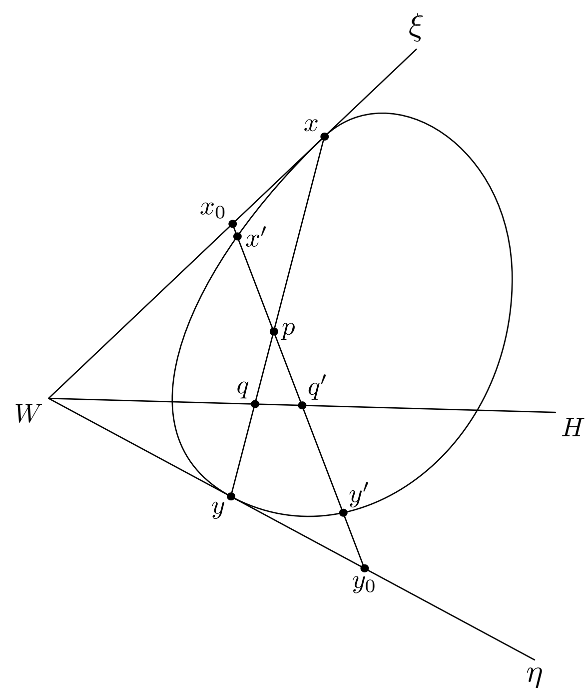

Figure 6. Metric projection of p on H.

Reciprocally: recall that when a point q' of Ω goes to the boundary, its distance to p goes to infinity. Hence by continuity of the distance and compactness there exists a point q on $H \cap \Omega$ such that $d_\Omega(p, H) = d_\Omega(p, q)$. Now consider the Hilbert ball $B_\Omega(p, r)$ of radius $r = d_\Omega(p, H)$ centred at p. Let once more x, y, ξ and η be defined as before, and let H' be the hyperplane passing by q and $\xi \cap \eta = W$. Then this hyperplane has to be tangent to the ball $B_\Omega(p, r)$, otherwise one can find a point q' on H' inside the open ball (i.e. $d(p, q') < r$), however by the reasoning done in our first step we would conclude that this point is at a distance bigger or equal to r, which would be a contradiction. By minimality of the point q, H is also a supporting hyperplane of

$B_\Omega(p,r)$ at q. Hence we have to distinguish between two cases. If Ω is C^1, then by uniqueness of the tangent hyperplanes at every point $H = H'$. Otherwise, Ω is not C^1 at x or y. In that case it is possible to change one of the hyperplane, say ξ, with ξ' passing by x and $H \cap \eta$ (which might be at infinity, which would mean that we consider parallel hyperplanes). $\square$

Notice that there is no uniqueness of the metric projections (also called "foot" by Busemann). However if the convex set is strictly convex, then we will have a unique projection, if furthermore the convex is C^1, this projection will be given by a unique pair of supporting hyperplanes.

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Constantin Vernicos Constantin.Vernicos@math.univ-montp2.fr
Institut de mathématique et de modélisation de Montpellier, Université Montpellier 2 , Case Courrier 051, Place Eugène Bataillon , F–34395 Montpellier Cedex, France