ASYMPTOTIC VOLUME IN HILBERT GEOMETRIES

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Abstract. We prove that the metric balls of a Hilbert geometry admit a volume growth at least polynomial of degree their dimension. We also characterise the convex polytopes as those having exactly polynomial volume growth of degree their dimension.

Introduction and statement of results

We recall that Hilbert geometries are metric space defined in the interior of a convex set using cross-ratios and as such are a generalisation of the hyperbolic geometry.

Among all Hilbert geometries, two families have emerged and play an important role. On the one hand the polytopal ones, which for a given dimension are all bi-lipschitz to the Hilbert geometry of the simplex ([BER09, Ver-a, CVV11]), and on the other hand those whose boundary is C^2 with positive Gaussian curvature, which are all bi-lipschitz to the Hyperbolic space ([CVc04]).

The present paper focuses on the volume of balls and was motivated by the following result due to Burago and Ivanov[BI95]:

Theorem 1. *Let (T^n, g) be a Riemannian torus, let ω_n be the euclidean volume of the euclidean unit ball, and let x be a point on the universal covering of T^n. Let also $B_g(x,r)$ be the metric ball of radius r of the lifted metric centred at x. Then*

- *Asvol(g)* $= \lim_{r\to+\infty} \dfrac{\mathrm{Vol}\big(B_g(x,r)\big)}{r^n} \geq \omega_n$;
- *Equality characterizes flat tori.*

The author belief is that a similar statement may exists in Hilbert geometry, the equality case characterising the simplexes. With that goal in mind we obtained a partial answer to that question and a new characterisation of polytopes in term of their volume growth as follows:

Theorem 2. *There exists a constant $c_n > 0$ such that for all Hilbert geometries $(\mathcal{C}, d_{\mathcal{C}})$, with $\mathcal{C} \subset \mathbb{R}^n$ a n-dimensional convex body, for any point $p \in \mathcal{C}$ and any real number $r > 0$, if one denotes by $\mathrm{Vol}_{\mathcal{C}}$ the Holmes-Thompson volume of $\mathcal{C}$, then we have*

- $\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x,r)\big) \geq c_n r^n$;

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- *The asymptotic volume is finite if and only if $\mathcal{C}$ is a polytope;*
- *If $\mathcal{C}$ is a polytope with k verticies, then one has*

$$\underline{\mathrm{Asvol}}(\mathcal{C}) = \liminf_{r\to+\infty} \frac{\mathrm{Vol}_{\mathcal{C}}(B_{\mathcal{C}}(x,r)}{r^n} \geq c_n k.$$

Therefore, in each dimension, there is only a finite number of families of polytopal Hilbert geometries which may have an asymptotic volume less than the simplex's.

This theorem is therefore weaker in its asymptotic results from the one we expect, but in the meantime it gives the existence of an optimal lower bound on the volume growth of balls which was not known. Indeed the previous result of this kind was obtained by the author with Colbois [CVs07], but it only gave a lower bound which converged to zero as the radius of the ball went to infinity.

As a corollary we obtain a new proof of the fact that a n-dimensional Hilbert geometry which quasi-isometrically embeds into a n-dimensional normed vector space is actually a polytopal one.

1. Notations

A *proper* open set in $\mathbb{R}^n$ is a set not containing a whole line.

A Hilbert geometry $(\mathcal{C}, d_{\mathcal{C}})$ is a non empty *proper* open convex set $\mathcal{C}$ in $\mathbb{R}^n$ (that we shall call *convex domain*) with the Hilbert distance $d_{\mathcal{C}}$ defined as follows: for any distinct points p and q in $\mathcal{C}$, the line passing through p and q meets the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ at two points a and b, such that one walking on the line goes consecutively by a, p, q, b. Then we define

$$d_{\mathcal{C}}(p,q) = \frac{1}{2}\ln[a,p,q,b],$$

where $[a,p,q,b]$ is the cross ratio of (a,p,q,b), i.e.,

$$[a,p,q,b] = \frac{\|q-a\|}{\|p-a\|} \times \frac{\|p-b\|}{\|q-b\|} > 1,$$

with $\|\cdot\|$ the canonical euclidean norm in $\mathbb{R}^n$. If either a or b is at infinity the corresponding ratio will be taken equal to 1.

Note that the invariance of the cross ratio by a projective map implies the invariance of $d_{\mathcal{C}}$ by such a map.

These geometries are naturally endowed with a C^0 Finsler metric $F_{\mathcal{C}}$ as follows: if $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^n$ with $v \neq 0$, the straight line passing by p and directed by v meets $\partial\mathcal{C}$ at two points $p_{\mathcal{C}}^+$ and $p_{\mathcal{C}}^-$. Then let t^+ and t^- be two positive numbers such that $p + t^+v = p_{\mathcal{C}}^+$ and $p - t^-v = p_{\mathcal{C}}^-$, in other words these numbers corresponds to the time necessary to reach the boundary starting at p with the speed v and $-v$. Then we define

$$F_{\mathcal{C}}(p,v) = \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad \text{and} \quad F_{\mathcal{C}}(p,0) = 0.$$

Should $p_{\mathcal{C}}^{+}$ or $p_{\mathcal{C}}^{-}$ be at infinity, then corresponding ratio will be taken equal to 0.

The Hilbert distance $d_{\mathcal{C}}$ is the length distance associated to $F_{\mathcal{C}}$. We shall denote by $B_{\mathcal{C}}(p,r)$ the metric ball of radius r centred at the point $p \in \mathcal{C}$.

Thanks to that Finsler metric, we can build two important Borel measures $\mathcal{C}$.

The first one is called the *Busemann* volume, will be denoted by $\mathrm{Vol}_{\mathcal{C}}$ (It is actually the Hausdorff measure associated to the metric space $(\mathcal{C}, d_{\mathcal{C}})$, see [BBI01], example 5.5.13), and is defined as follows. To any $p \in \mathcal{C}$, let $\beta_{\mathcal{C}}(p) = \{v \in \mathbb{R}^n \mid F_{\mathcal{C}}(p,v) < 1\}$ be the open unit ball in $T_p\mathcal{C} = \mathbb{R}^n$ of the norm $F_{\mathcal{C}}(p,\cdot)$ and ω_n the euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^n$. Consider the (density) function $h_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathbb{R}$ given by $h_{\mathcal{C}}(p) = \omega_n/\mathrm{Leb}\big(\beta_{\mathcal{C}}(p)\big)$, where Leb is the canonical Lebesgue measure of $\mathbb{R}^n$ equal to 1 on the unit "hypercube".

$$\mathrm{Vol}_{\mathcal{C}}(A) = \int_A h_{\mathcal{C}}(p) \mathrm{dLeb}(p)$$

for any Borel set A of $\mathcal{C}$.

The second one, called the *Holmes-Thompson* volume will be denoted by $\mu_{HT,\mathcal{C}}$, and is defined as follows. Let $\beta_{\mathcal{C}}^{*}(p)$ be the polar dual of $\beta_{\mathcal{C}}(p)$ and $h_{HT,\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathbb{R}$ the density defined by $h_{HT,\mathcal{C}}(p) = \mathrm{Leb}\big(\beta_{\mathcal{C}}^{*}(p)\big)/\omega_n$. Then $\mu_{HT,\mathcal{C}}$ is the measure associated to that density.

We can actually consider a wider family of measure, including the Holmes-Thompson and the Buseman ones, as follows. Let $\mathcal{E}_n$ be the set of pointed properly open convex sets in $\mathbb{R}^n$. These are the pairs (ω, x), such that ω is a properly open convex set and x a point inside ω. We shall say that a function $f \colon \mathcal{E}_n \to \mathbb{R}^+ \setminus \{0\}$ is a *proper density* if it is

Continuous: with respect to the Hausdorff pointed topology on $\mathcal{E}_n$;

Monotone decreasing: with respect to inclusion of the convex sets, i.e., if $x \in \omega \subset \Omega$ then $f(\Omega, x) \leq f(\omega, x)$.

Chain rule compatible: if for any projective transformation T one has

$$f\big(T(\omega), T(x)\big)\mathrm{Jac}(T) = f(\omega, x).$$

We will say that f is a *normalised proper density* if in addition f coincides with the standard Riemannian volume on the Hyperbolic geometry of ellipsoids. Let us denote by PD_n the set of proper densities over $\mathcal{E}_n$.

Let us now recall a result of Benzécri [Ben60] which states that the action of the group of projective transformations on $\mathcal{E}_n$ is co-compact. Then, as remarked by L. Marquis, for any pair f, g of proper densities, there exists a constant $C > 0$ ($C \geq 1$ for the normalised ones) such that that for any $(\omega, x) \in \mathcal{E}$ one has

$$\frac{1}{C} \leq \frac{f(\omega, x)}{g(\omega, x)} \leq C. \tag{1}$$

In the same way we defined the Busemann and the Holmes-Thompson volumes, to any proper density f one can associate a Borel measure $\mu_{f,\mathcal{C}}$ on $\mathcal{C}$. Integrating the equivalence (1) we obtain that for any pair f, g of densities, there exists a constant $C > 0$ such that for any Borel set $U \subset \mathcal{C}$ we will have

$$\frac{1}{C}\mu_{g,\mathcal{C}}(U) \leq \mu_{f,\mathcal{C}}(U) \leq C\mu_{g,\mathcal{C}}(U). \tag{2}$$

We shall call *proper measures with density* the family of measures obtain this way.

To a proper density $f \in PD_{n-1}$ we can also associate a $n-1$-dimensional measure, denoted by $\text{Area}_{f,\mathcal{C}}$, on hypersurfaces in $\mathcal{C}$ as follows. Let S_{n-1} be smooth a hypersurface, and consider for a point p in the hypersurface S_{n-1} its tangent hyperplane $H(p)$, then the measure will be given by

$$d\text{Area}_{f,\mathcal{C}}(p) = d\mu_{f,\mathcal{C}\cap H(p)}(p). \tag{3}$$

Let now $\mu_{f,\mathcal{C}}$ be a proper measure with density over $\mathcal{C}$, then the volume entropy of $\mathcal{C}$ is defined by

$$\text{Ent}(\mathcal{C}) = \liminf_{r\to+\infty} \frac{\ln \mu_{f,\mathcal{C}}\big(B_{\mathcal{C}}(p, r)\big)}{r}. \tag{4}$$

This number does not depend on either f or p.

2. Lower bound

Theorem 3. *The volume growth of balls and spheres in a Hilbert geometry is at least polynomial. More precisely, for any integer $n \in \mathbb{N}^*$ and any proper density $f \in PD_n$ (resp. $g \in PD_{n-1}$) there exists a constant $c_B(n, f)$ (resp. $c_S(n, g)$) such that given a n-dimensional Hilbert Geometry $(\mathcal{C}, d_{\mathcal{C}})$ and a point $x \in \mathcal{C}$, for any $r \in \mathbb{R}^+$ one has the following inequalities*

$$\begin{aligned} c_B(n, f)r^n &\leq \mu_{f,c}\big(B_{\mathcal{C}}(x, r)\big) \\ c_S(n, g)r^{n-1} &\leq Area_{g,\mathcal{C}}\big(S_{\mathcal{C}}(x, r)\big). \end{aligned}$$

When considering the Busemann volume we will drop the f or g in the constants, i.e., we shall just denote the constants appearing in Theorem 3 by $c_B(n)$ and $c_S(n)$. The constant associated to the Holmes-Thompson metric will be denoted by $c_B^*(n)$ and $c_S^*(n)$.

Proof. According to the inequality (2) there exists a constant $C_1(n)$ such that one has the following comparison between the Holmes-Thompson and the Busemann measures: for any Borel set U in $\mathcal{C}$,

$$C_1^{-1}(n)\mu_{HT,\mathcal{C}}(U) \leq \mathrm{Vol}_{\mathcal{C}}\big(U\big) \leq C_1(n)\mu_{HT,\mathcal{C}}(U). \tag{5}$$

hence our results will be true for either of these measures (and actually for any proper density).

Remark that without loss of generality we can restrict to *strongly* convex sets, *i.e.*, convex sets with C^2 boundary and stritcly positive Gaussian curvature, as the results passes to the limit with respect to the Hausdorff pointed topology.

Now let us do the proof by induction on the dimension for both measures at the same time. First notice that the 1-dimensional Hilbert geometry is isometric to $\mathbb{R}$ thus, $c_B(1) = 2 = c_B^*(1)$ and we have actually an equality for both measures.

Now suppose the result is true in dimension n and let us prove that it holds in dimension $n+1$. We need to consider a point $x \in \mathcal{C} \subset \mathbb{R}^{n+1}$ and the ball of radius r centred at x. Take a hyperplane H intersecting the convex set $\mathcal{C}$ and containing x, by induction we thus have for any $s \in \mathbb{R}^+$

$$c_B^*(n)s^n \leq \mu_{HT,\mathcal{C}\cap H}(B_{\mathcal{C}\cap H}(x,s))$$

and we remark that $\mathcal{C}\cap H$ is totally geodesic, thus thanks to a Crofton formula valid in this setting (see [AF98] Theorem 1.1 and Remark 2) or by minimality of totally geodesic submanifolds with respect to the Holmes-Thompson measure (see [AB09, Ber09]) we obtain that the Holmes-Thompson area of the half spheres of radius s centred at x defined by H have an area bigger or equal to $B_{\mathcal{C}\cap H}(x,s)$, hence

$$2c_B^*(n)s^n \leq \mathrm{Area}_{HT,\mathcal{C}}\big(S_{\mathcal{C}}(x,s)\big)$$

Which implies the result for the spheres. Now thanks to the comparison between the Holmes-Thompson and Busemann volume given by equation (5) and the the co-area inequality obtained in [BBV10] (lemma 2.13) we have the existence of a constant $C_2(n)$ such that

$$2c_B^*(n)s^n \leq \mathrm{Area}_{HT}\big(S_{\mathcal{C}}(x,s)\big) \leq C_2(n)\frac{\partial}{\partial s}\mathrm{Vol}\big(B_{\mathcal{C}}(x,s)\big) \tag{6}$$

Hence it suffices to integrate the inequalities (6) between 0 and r to obtain the desired result for the Busemann measure and thanks to the comparison (5) for the Holmes-Thompson measure. □

The previous proof also implies the following proposition related to the volume entropy.

Proposition 4. *The volume entropy of a Hilbert geometry is bigger or equal to any of its lower dimensional sections, i.e., let* $(\mathcal{C}, d_{\mathcal{C}})$ *be a*

n-dimensional Hilbert geometry and let A_k be an affine k-dimensional subspace of $\mathbb{R}^n$, then we have

$$\mathrm{Ent}(A_k \cap \mathcal{C}) \leq \mathrm{Ent}(\mathcal{C}).$$

Proof. We just do the proof for $k = n-1$, the general result easily follows. Let H be a hyperplane such that $H \cap \mathcal{C}$ is an open $(n-1)$-dimensional convex set and p be a point inside $H \cap \mathcal{C}$. Then by minimality, as in the previous proof one has

$$2\mu_{HT,H\cap\mathcal{C}}\big(B_{H\cap\mathcal{C}}(p,R)\big) \leq \mathrm{Area}_{HT,\mathcal{C}}\big(S_{\mathcal{C}}(p,R)\big).$$

Now taking the logarithm of both sides, dividing by R and taking the limit as R goes to infinity proves that $\mathrm{Ent}(H\cap\mathcal{C})$ is lower than the spherical volume entropy of $\mathcal{C}$, which is equal to the volume entropy of $\mathcal{C}$ following [BBV10]. □

Let us now focus on the Busemann volume and define the n-asymptotic volume by

$$\underline{\mathrm{Asvol}_n}(\mathcal{C},x) = \liminf_{r\to+\infty} \frac{\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x,r)\big)}{r^n} \tag{7}$$

Conjecture 1. *Let b_n be the asymptotic volume of the simplex (which equals the euclidean volume of the unit ball if the volume is the Busemann volume) then we have*

(1) $\underline{\mathrm{Asvol}_n}(\mathcal{C}) \geq b_n$;
(2) with equality if and only if $(\mathcal{C}, d_{\mathcal{C}})$ is a simplex.

In a previous paper we studied the volume entropy of Hilbert geometries [BBV10]. In the present paper we are focusing on Hilbert geometries for which the entropy is equal to zero.

In that case one can focus on the polytopal entropy defined by

$$\mathrm{PolEnt}(\mathcal{C}) = \liminf_{r\to+\infty} \frac{\ln\Big(\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x,r)\big)\Big)}{\ln r}. \tag{8}$$

This number can be defined for any proper measure with density, and does not depend on the proper density nor on the centre x.

3. Upper Bound

In this section we will consider the Busemann volume and denote once more by $c_B(n)$ the constant given by Theorem 3.

Let us define the upper n-asymptotic volume by

$$\overline{\mathrm{Asvol}_n}(\mathcal{C},x) = \limsup_{r\to+\infty} \frac{\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x,r)\big)}{r^n} \tag{9}$$

Proposition 5. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be an n-dimensional Hilbert geometry, the upper n-asymptotic volume is finite, if and only if $\mathcal{C}$ is a polytope, i.e.,*

$$\overline{\mathrm{Asvol}_n}(\mathcal{C}) = \limsup_{r\to+\infty} \frac{\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x,r)\big)}{r^n} < +\infty \iff \mathcal{C} \text{ is a polytope}$$

Remark 6. The results obtained in Vernicos [Ver-b] give for any $n \in \mathbb{N}$ the existence of convex sets in $\mathbb{R}^2$ with polynomial volume growth, such that $n+2 \leq \mathrm{PolEnt}(\mathcal{C}) \leq n+3$ which are therefore not polytopes.

Let us start with two easy lemmata which play a crucial role in the proof of proposition 5.

Lemma 7. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a two dimensional Hilbert geometry and p, q two extremal points on $\partial\mathcal{C}$ admitting supporting lines disjoint from the line (pq). Let x be any point in $\mathcal{C}$ and let $x_p(R)$, $x_q(R)$ be the intersection of the lines (xp) and (xq) with the sphere of radius R centred at x, i.e., $S_{\mathcal{C}}(x,R)$, then one has*

$$\lim_{R\to+\infty} \frac{d_{\mathcal{C}}\big(x_p(R), x_q(R)\big)}{2R} = 1.$$

Proof. The triangle inequality implies that

$$d_{\mathcal{C}}\big(x_p(R), x_q(R)\big) \leq 2R. \tag{10}$$

It remains to bound from below this number by a function converging to $2R$ as $R \to +\infty$.

To do so, let H_p and H_q the two disjoint supporting lines of $\mathcal{C}$ respectively at p and q, defined thanks to the affine functions f_p and f_q, which are supposed to be strictly positive on $\mathcal{C}$. Let $\mathcal{S}_{p,q}$ be the convex set defined by $\mathcal{S}_{p,q} = \{z \mid f_p(z) > 0 \text{ and } f_q(z) > 0\}$, by definition it contains $\mathcal{C}$. Then we have the usual comparison, even if the convex is not bounded,

$$d_{\mathcal{C}}\big(x_p(R), x_q(R)\big) \geq d_{\mathcal{S}_{p,q}}\big(x_p(R), x_q(R)\big). \tag{11}$$

Now by definition of the Hilbert metric and of the point $x_p(R)$ we have

$$d_{\mathcal{S}_{p,q}}\big(x, x_p(R)\big) \sim \frac{1}{2}\ln\frac{||x-p||}{||x_p(R)-p||} \sim d_{\mathcal{C}}\big(x, x_p(R)\big) = R$$

and the same equivalence holds when replacing p by q. Now, denote by $p(R)$ and $q(R)$ the intesection of the line $\big(x_p(R)x_q(R)\big)$ respectively with H_p and H_q. Then

$$d_{\mathcal{S}_{p,q}}\big(x_p(R), x_q(R)\big) = \frac{1}{2}\ln\left(\frac{||p(R)-x_q(R)||}{||p(R)-x_p(R)||}\right) + \frac{1}{2}\left(\frac{||q(R)-x_p(R)||}{||q(R)-x_q(R)||}\right)$$

Hence, thanks to the inequalities (10) and (11) in order to conclude it suffices to show that

$$\ln\left(\frac{||p(R)-x_q(R)||}{||p(R)-x_p(R)||}\right)\sim \ln\frac{||x-p||}{||x_p(R)-p||}, \tag{12}$$

and the same when commuting p and q. But as p and q play similar roles we just need to do the computations for one of them.

Let $A(R)$ be the intersection of the line (xp) with the line through $x_q(R)$ parallel to H_p (i.e. the line whose equation is $f_p(z)=f_p\big(x_q(R)\big)$). By Thales's Theorem we have

$$\frac{||x_p(R)-x_q(R)||}{||p(R)-x_p(R)||}=\frac{||x_p(R)-A(R)||}{||p-x_p(R)||} \tag{13}$$

Notice that as $R\to+\infty$, $A(R)$ converges to $A(\infty)$, the intersection between (xp) and the line through q parallel to H_p, let us denote by $K=||p-A(\infty)||$. By assumption H_p is different from (pq) which implies that $A(\infty)$ is not p, hence $||p-A(\infty)||>0$. Therefore for R large enough, $||x_p(R)-A(R)||$ is bounded from above, let say by $2K$, and from below by $K/2$.

To finish the proof of the equivalence (12) let us notice that

$$\begin{aligned}\ln\left(\frac{||p(R)-x_q(R)||}{||p(R)-x_p(R)||}\right) &= \ln\left(1+\frac{||x_p(R)-x_q(R)||}{||p(R)-x_p(R)||}\right)\\ &= \ln\left(1+\frac{||x_p(R)-A(R)||}{||p-x_p(R)||}\right)\text{by (13)}\\ &= \ln\left(\frac{||x-p||}{||p-x_p(R)||}\right)\\ &\quad+\ln\left(\frac{||p-x_p(R)||+||x_p(R)-A(R)||}{||x-p||}\right),\end{aligned}$$

the second part of this last equality converges to $\ln(K/||x-p||)$ and the first part to $2R$.

□

Lemma 8. *Let $(\mathcal{C},d_{\mathcal{C}})$ be a two dimensional Hilbert geometry. Then $(\mathcal{C},d_{\mathcal{C}})$ is a polygone if and only if the family of extreme points on its boundary such that any two of them satisfy the assumptions of lemma 7 is finite.*

Proof. Suppose that $(\mathcal{C},d_{\mathcal{C}})$ satisfies the assumption and is not a polygone. By Krein-Millman's theorem there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of disjoint extremal points of $\partial\mathcal{C}$. However, for a given point x_k and a given support line l_k at x_k there is at most one other point $x_{k'}$ which admits also l_k as a support line. Hence we can extract a subsequence $(x_{\varphi(k)})_{k\in\mathbb{N}}$ and a family H_k of distincts support lines, none of them containing, any two points of $(x_{\varphi(k)})_{k\in\mathbb{N}}$, which contradicts our assumption. □

Proof of Proposition 5. In the proof of Proposition 4 in[Ver09] we proved that in a polytope of $\mathbb{R}^n$, the volume of balls of radius r was less than a constant times r^n. Once again the co-area inequality obtained in [BBV10] implies that the volume of spheres of radius r is also bounded by a constant times r^{n-1}, otherwise we would get a contradiction.

Reciprocally, let us suppose that $\mathcal{C}$ is not a polytope. Then it admits a plane section which is not a polygon, and by lemma 8 on the boundary of that section for any $k \in \mathbb{N}^*$ there exist a subset X_k with k extremal points of $\partial\mathcal{C}$ such that any two of them satisfy the assumptions of Lemma 7. Without loss of generality, wa can assume that x belongs to that plane section.

Then let us fix some k and a corresponding subset X_k of the boundary. Then following Lemma 7 for any given pair of points (p, q) in X_k, there exists an $R_{p,q} > 0$ such that for any $R > R_{p,q}$, if $x_p(R)$ and $x_q(R)$ are the intersections of the lines (xp) and (xq) with the sphere of radius R centred at x, *i.e.*, $S_{\mathcal{C}}(x, R)$, then

$$d_{\mathcal{C}}\big(x_p(R), x_q(R)\big) > R.$$

Let us consider $R_k > \max\{R_{p,q} \mid p, q \in X_k\}$

For any extremal point $p \in X_k$ let us consider the ball $B_{p,R}$ of radius $R/4$ centred at $x_p(3R/4)$. Then for any radius $R > R_k$, and any pair of points $p, q \in X_k$, the corresponding balls $B_{p,R}$ and $B_{q,R}$ are disjoints. Thus

$$\coprod_{p \in X_k} B_{p,R} \subset B_{\mathcal{C}}(x, R). \tag{14}$$

Hence for any $R > R_k$ the sum of the volume of the balls $B_{p,R}$, for $p \in X_k$, is smaller than the volume of the ball of radius R centred at x, i.e.,

$$\sum_{p \in X_k} \mathrm{Vol}_{\mathcal{C}}\big(B_{p,R}\big) \leq \mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x, R)\big). \tag{15}$$

We now apply the lower bound on the volume of the balls of radius $R/4$ obtained in Theorem 3 to the inequality (15) to obtain a lower bound in terms of k and R^n:

$$k c_B(n)(R/4)^n \leq \mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x, R)\big), \tag{16}$$

and taking the limit as R goes to infinity we finally get

$$k\frac{c_B(n)}{4^n} \leq \underline{\mathrm{Asvol}_n}(\mathcal{C}). \tag{17}$$

This being true for any integer $k \in \mathbb{N}$ we conclude that $\underline{\mathrm{Asvol}_n}(\mathcal{C})$ is infinite. □

During the previous proof with ended up with the equation (17) which can be summed up in the following way.

Proposition 9. *For any integer $n \in \mathbb{N}$, there exists a constant $a(n)$ such that for any polytope $\mathcal{P}_k$ with k vertices and non-empty interior in $\mathbb{R}^n$ one has*

$$a(n)k \leq \underline{\mathrm{Asvol}_n}(\mathcal{C}).$$

We also get the following corollary, which is also a consequence of Colbois-Verovic [CVc11]

Corollary 10. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert geometry in $\mathbb{R}^n$ which quasi-isometrically embeds into a n-dimensional vector space, then $\mathcal{C}$ is a polytope.*

Proof. Let us denote the vector space by $(V, \|\cdot\|)$, then by definition, this means that there exists a function $f\colon \mathcal{C} \to \mathbb{R}^n$ and two constants A and B such that

$$\frac{1}{A} d_{\mathcal{C}}(p,q) - B \leq \big\|f(p) - f(q)\big\| \leq A d_{\mathcal{C}}(p,q) + B.$$

Now consider a ball of radius $R > 2AB$ centred at x in $(\mathcal{C}, d_{\mathcal{C}})$ and consider a maximal $2BA$ separated set $\mathfrak{S}$ in that ball. Let p, q be two points in that set, then $\big\|f(p) - f(q)\big\| \geq B$, hence this is a B separated set in V.

Moreover the image of the ball of radius R centred at x is included in the ball of radius $AR + B$ centred at $f(x)$.

Therefore, using the volume of ball in V, we deduce that $\mathfrak{S}$ has less than $(AR + B)^n / B^n$ points.

Recall that, following Colbois-Vernicos [CVs06] (see Théorème 9), there is a constant $V(2AB)$ independant of the n-dimensional Hilbert geometry such that for any point y, $\mathrm{Vol}_{\mathcal{C}}(B_{\mathcal{C}}(y, 2AB) \leq V(2AB)$. As a maximal $2AB$-separated set is also $2AB$ covering, *i.e.* any other point is at distance less than or equal to $2AB$ from $\mathfrak{S}$, we obtain that

$$\mathrm{Vol}_{\mathcal{C}}\big(B_{\mathcal{C}}(x, R)\big) \leq V(2AB)\frac{(AR + B)^n}{B^n}$$

which implies that the upper asymptotic volume of $\mathcal{C}$ is finite. □

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