CIRM
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Small Workshop
on Hilbert Geometries

Some open problems related to Hilbert geometries

Contribution by J.-C. Àlvarez-Paiva

Question 1 : Give conditions on a finite metric space with n points to admit (or not to admit) an isometric embedding into a Hilbert geometry. Note that you can easily 4-point spaces that do not embed in a hyperbolic space of any dimension. One may play with the dimension of the Hilbert geometry as well: does every 4-point metric space admit an isometric embedding into some 2-dimensional Hilbert geometry? How about 5-point spaces? You get the idea ...

More Gromov-ically: consider the convex cone of metrics (distance functions) defined on n points. Describe the subset of this cone defined by the set of embeddings of n points into any Hilbert geometry.

1.a Comment by Bas Lemmens

A related problem is to determine the equilateral dimension of a Hilbert geometry (X, d_X), which is defined to be the maximum number of points in X that are all at the same distance from each other. This problem has been extensively studied by C. Swanepoel and others in finite dimensional ℓ_p-spaces and appears to be hard.

Question 2 : Assume all hyperplanes in a Hilbert geometry are minimal hypersurfaces for the Hausdorff area functional. Does that mean that the Hilbert geometry is just hyperbolic geometry? I asked this on "What's wrong with the Hausdorff measure ..."

Question 3 : Study the isoperimetric inequality and related questions in Hilbert geometries. I have seen nothing at all on this topic. Maybe it's impossibly hard, maybe no one has looked at it seriously. There are the results of Wenger on isoperimetric inequalities and Gromov-hyperbolic spaces, but one should look for something sharper and use the Holmes-Thompson volume.

Question 4 : Forget about compact quotients of Hilbert geometries (for a while) and study the more general class of compact (reversible and non-reversible) Finsler surfaces of genus greater than one without focal points. Here I define "without focal points" in a geometric way—metric balls in the universal covering are convex—and allow low regularity Finsler spaces or even Busemann's G-spaces. Can one prove that the geodesic flow is transitive for such a manifold ? What about entropy and the like?

Contribution by M. Crampon and L. Marquis

Question 5 : In dimension 2, Hilbert metric can be defined through a measure on the set of geodesics and Crofton formula.

5.a Is there an equivalent of this fact in higher dimension? Is is true for all Hilbert metrics, even the most irregular ones? That should be a measure on sets of hyperplanes intersecting the convex set. Maybe Juan-Carlos can explain more about that. I guess more is known about the regular geometries, but what about the others? A positive answer to this question in the strictly convex case (or even just in the Gromov-hyperbolic case) should show that the automorphism group of a quasi-divisible convex (that is, it admits with a quotient of finite volume) has the Haagerup property. An interesting corollary will be, with the help of Kapovich's work, that the fundamental group of Gromov-Thurston manifolds has the Haagerup property.

5.b If Ω is strictly convex with a compact quotient M, this measure yields an invariant measure μ of the geodesic flow on HM, which is also flip-invariant. What are the dynamical properties of this measure? In the case Ω is an ellipsoid, one recovers in this way the Liouville measure, which is also the measure of maximal entropy. When Ω is not an ellipsoid, there is no absolutely continuous invariant measure, but the measure of maximal entropy is still defined and flip-invariant. Is there a rigidity result: μ is the measure of maximal entropy if and only if Ω is an ellipsoid? This question was asked to M.C. by Yong Fang, from the University of Cergy-Pontoise.

5.c Some comments on question 5 and a few additional questions by Juan-Carlos Alvarez-Paiva.

If the boundary of the convex body K is sufficiently smooth, then yes there is a (possibly signed) measure on the set of hyperplanes such that Crofton's formula holds. I proved this with Fernandes for all projective Finsler spaces and we also showed that the measure can be constructed from the Fourier transform—in the velocities—of the Finsler metric. The conditions on K that guarantee that this measure is never negative are not yet known. By a theorem of Ralph Alexander this is equivalent to asking conditions on the Hilbert geometry to be hypermetric (an important concept in metric geometry) or, equivalently, to asking when a Hilbert geometry admits an isometric embedding into the Banach space L_1. Here I mean a metric isometric embedding (distances are preserved) and not a Finsler isometric embedding (lengths of curves are preserved).

Question 5.c: For what convex bodies (in dimension > 2) is the associated Hilbert geometry hypermetric?

I asked myself this question when I was writing my survey on Finsler geometry, but then other people (Rolf Schneider among others) have asked me the same thing. It seems to be a "folk problem".

If the boundary of K is not very regular, then there may be no measure for which the Crofton formula is true, BUT there will always be a distribution in a certain class of distributions which will do the work. This has not really been well studied, but one should check out the work of Szabo on Hilbert's fourth problem. The main results of his paper are related to regularity issues in the integral-geometric solution of Hilbert's fourth problem.

In the second part of Question 5, Michael mixes things a bit and suddenly we're back on the space of geodesics instead on the space of hyperplanes. On the space of geodesics there is always a non-negative measure. Through the Crofton formula this measure can be used to compute the Holmes-Thompson area of hypersurfaces in the Hilbert geometry. Moreover, the Holmes-Thompson area functional in a reversible Finsler manifold completely determines the metric, so one can say that the measure on the space of geodesics determines the metric.

Here is something that always puzzles me when I hear talks on geodesic flows on compact quotients of Hilbert geometries: the flow is always defined in the tangent bundle, however the Legendre transformation from the tangent to the cotangent space is very bad. So bad in fact that the pullback of the Liouville measure is singular with respect to the Lebesgue measure on the unit tangent bundle. The pullback of the symplectic form is therefore very bad as well and we must conclude that this “geodesic flow" is not a Hamiltonian flow.

Is there a geometric way to see this other than the fact that there is no invariant measure for this flow in the Lebesgue class?

Question 6 : Yves Benoist characterized the Hilbert geometries which are Gromov-hyperbolic: they are those whose boundary is quasi-symmetrically convex. Given any Hilbert geometry (Ω, d_Ω), can we find such a condition for a convex subset A of the geometry to be Gromov-hyperbolic for the metric d_Ω?

Question 7 : Given a compact quotient M of a strictly convex Hilbert geometry (Ω, d_Ω) of dimension n, M.C. proved that the sum χ^+ of positive Lyapunov exponents, counted with multiplicities, is $n-1$, as for an ellipsoid. Is there a rigidity result: all Lyapunov exponents are equal if and only if Ω is an ellipsoid? This question was asked to M.C. by Patrick Foulon during his Ph.D.

Question 8 : Do there exist exotic finite volume quotients, as constructed by Yves Benoist or Misha Kapovich in the compact case? That is, we look for a finite volume quotient $M = \Omega/_\Gamma$ such that Ω is strictly convex but not quasi-isometric to the hyperbolic space.

Question 9 : Is it possible to deform Benoist and Kapovich exotic structures ?

Question 10 : Given an irreducible discrete subgroup Γ of $\mathrm{Aut}(\Omega)$, consider its critical exponent

$$\delta_\Gamma = \limsup_{R \to +\infty} \frac{1}{R} \log \sharp\{g \in \Gamma,\ d_\Omega(x, gx) \leqslant R\}.$$

The limit set Λ_Γ of Γ is the closure of the set of attractive points of proximal elements. Its convex hull $C(\Lambda_\Gamma)$ is the smallest open convex set Γ can act on. Do we have

$$\delta_\Gamma = h_{vol}(C(\Lambda_\Gamma), d_{C(\Lambda_\Gamma)})?$$

It is known to be true (Crampon-Marquis) if Ω is strictly convex (with $\mathcal{C}^1$ boundary) and $M = \Omega/_\Gamma$ has finite volume.

Question 11 : Can we characterize the subgroups Γ of $\mathrm{SL}_{n+1}(\mathbb{R})$ which preserve a strictly convex (with $\mathcal{C}^1$-boundary) Hilbert geometry? If Γ preserves a strictly convex set, it also preserves the dual convex set whose boundary is $\mathcal{C}^1$. Is it true that, if Γ preserves a strictly convex set, then one can find a Γ-invariant strictly convex set with $\mathcal{C}^1$ boundary?

Question 12 : Can we find a geometrical/dynamical description of the boundary of the deformation space $\mathcal{G}(\Sigma_g)$ of convex projective structures on a surface Σ_g of genus $g \geqslant 2$? I guess the only description is Anne Parreau's one in terms of the action of the fundamental group on the symmetric space $\mathrm{SL}_{n+1}(\mathbb{R})/_{\mathrm{SO}_{n,1}(\mathbb{R})}$.

Question 13 : M.C. proved the entropy function $h : \mathcal{G}(\Sigma_g) \longrightarrow \mathbb{R}$ takes its values in $(0, 1]$ and Xin Nie proved it is surjective in $(0, 1]$. Given any $x \in [0, 1]$, can we find a diverging sequence of structures whose entropy tends to x ? Nie proved the result for $x = 0$ and it is trivial for $x = 1$ since it is the entropy of a hyperbolic space.

Question 14 : Describe the moduli space of convex projective structures on a circle-bundle on a surface. This question should be link to question 12.

Contribution by S. Gaubert

Question 15 : For which integers $k > 0$ is there a unique ball of minimal radius containing those points in a Hilbert geometry in $\mathbb{R}^n$ (see also the contribution by J. Lawson).

Contribution by B. Lemmens

Question 16 : Let (X, d_X) be a Hilbert geometry and $f\colon X \to X$ be a non-expansive map on X, so $d_X(f(x), f(y)) \leq d_X(x,y)$ for all $x, y \in X$. Alan Beardon proved that if X is a strictly convex domain and f has **no** fixed point in X, then there exists $\eta \in \partial X$ such that $\lim_{k\to\infty} f^k(x) = \eta$ for all $x \in X$, and the convergence is uniform on compact subsets of X. What about general Hilbert geometries?

It was conjecture independently by Anders Karlsson and Roger Nussbaum that if (X, d_X) is a general Hilbert geometry and f is a fixed point free non-expansive map on X, then there exists a convex subset $\Lambda \subset \partial X$ such that for all $x \in X$ each limit point of the orbit $\mathcal{O}_x = \{f^k(x)\colon k = 0, 1, 2, \ldots\}$ lies inside Λ. For polyhedral domains this conjecture was confirmed by Brain Lins in "A Denjoy-Wolff theorem for Hilbert metric nonexpansive maps on a polyhedral cone", Math. Proc. Camb. Phil. Soc., 143 (2007), 157–164.

Question 17 : It is well-known that the Hilbert geometry on a n-simplex is isometric to $(\mathbb{R}^n, \|\cdot\|_H)$, where

$$\|x\|_H = (\max_i x_i) \vee 0 - (\min_i x_i) \wedge 0$$

is a polyhedral norm. In general, if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a non-expansive map under a norm with polyhedral unit ball, and f has a fixed point, then every orbit of f converges to a periodic orbit of f, i.e., for each x there exists an integer p such that $f^{kp}(x) \to \xi_x$ as $k \to \infty$, where ξ_x is a periodic point of f with period p. Moreover there exists an a-priori upper bound for maximum period of periodic points of f in terms of the number of facets of the unit ball of the norm. Thus, if f is a Hilbert metric non-expansive map on the n-simplex and f has a fixed point, then every orbit converges to a periodic orbit of f, and there exists an a-priori upper bound for the maximum period of its periodic points in terms of n. What is the optimal upper bound? For the 2-simplex it was shown by Lemmens (Nonexpansive mappings on Hilbert's metric spaces, Topological Methods in Nonlinear Analysis, 38(1), (2011), 45–58) that 6 is the optimal upper bound. The current best upper bound is:

$$\max_k 2^k \binom{m}{k},$$

where $m = n(n+1)/2$. It is believed, however, that there exists $c > 2$ such that the periods of the periodic points do not exceed c^n, but no proof is known.

Question 18 : The following conjecture was posed by Bletz-Siebert and Foertsch in "Euclidean rank of Hilbert geometries", Pacific J. Math. 231(2), (2007), 257–278. It is impossible to isometrically embed a Euclidean plane into any Hilbert geometry (X, d_X), or, as they say the Euclidean rank of any Hilbert geometry is 1. They proved the conjecture for all strictly convex domains X, for all domains whose boundary is C^1 smooth, and for all 3-dimensional domains. It is also known to be true for polyhedral domains X, as it is impossible to have a non-expansive map $f\colon S \to S$, with $S \subseteq X$, which has a fixed point and whose orbits do not converge to periodic orbits.

A variant of this problem was posed by Cormac Walsh. Walsh defined the Minkowski rank of Hilbert geometry (X, d_X) as the largest dimension of a finite dimensional normed space that can be isometrically embedded into (X, d_X). As the Hilbert on the n-simplex is isometric to an n-dimensional normed space its Minkowski rank is n. It also easy to show that the Minkowski rank of any strictly convex Hilbert geometry is 1. In general, Walsh conjectured that the Minkowski rank of a Hilbert geometry (X, d_X) is the maximum dimension of a affine subspace H such that $H \cap X$ is a simplex.

Contrinution by J. Lawson

Question 19 : Is there a reasonable notion of the centroid (barycenter) of n points in a Hilbert geometry ?

Contribution by A. Papadopoulos

Question 20 : Find a properly open convex set in a finite or infinite-dimensional Banach space which contains naturally Teichmüller space. See Oberwolfach December 2010, see: Problem 13 in Problem Session Teichmüler Theory, Oberwolfach Reports Vol. 7, (2010) Issue 4. Study the induced Hilbert metric on the Teichmüller space. Compare it to the other known metrics. It seems that there is a beginning of an answer on ArXiV recently. One good answer was given recently by Sumio Yamada, in relation with Weil-Petersson geometry,see http://arxiv.org/pdf/1110.5022v1. Yohei Komori told Athanase (orally) about another approach.

Question 21 : There are two natural Riemmanian metrics on a properly convex open set Ω. Both are built using the convex hypersurface living in the cone above Ω, one using Vinberg characteristic function, the second one using the theory of affine spheres. The first one is analytic, the second one $\mathcal{C}^\infty$. We call the first one the analytic metric, the second one the affine metric. Benzécri's theorem show that there are two constants a_{analy} (resp. a_{aff}) and b_{analy} (resp. b_{aff}) such that the curvature of the analytic metric satisfy $b_{analy} \leqslant \kappa \leqslant a_{analy}$, and the analogous for the affine metric. It is easy to see that a_{aff} and a_{analy} are $\geqslant 0$ and that b_{aff} and b_{analy} are $\leqslant -1$. Can we compute precisely these constants?

Contribution by M. Troyanov

Question 22 : "Metaquestion": What can be said about the group of isometry $\text{Isom}(\Omega)$ of the Hilbert geometry of a convex set Ω, bounded or not.

22.a Compare it to the group of collineation, $\text{Coll}(\Omega)$.

22.b When is $\text{Isom}(\Omega)$ transitive ?

22.c What can be said about $\text{Isom}(\Omega_1 \times \Omega_2)$?

22.d When does $\text{Isom}(\Omega)$ contains a transitive abelian group $\mathbb{R}^n$?

22.e When is $\text{Isom}(\Omega) = \text{bounded} \times \mathbb{R}^k$?

22.f When is $\mathrm{Isom}(\Omega)$ a Lie group ?

NOTE BY C. VERNICOS: During the problem session the question of wether "not bounded" was important was raised. Indeed we usually study Hilbert geometries which are properly open convex set, which means that in some affine chart they are bounded.

Contribution by C. Vernicos

Question 23 : Regarding the bottom of the spectrum and amenability

23.a One of the first question B. Colbois told me about, was to know if the bottom of the spectrum (which can be define thanks to Raleigh quotient despite any definition of a Laplacian) of a Hilbert Geometry was always less than $(n-1)^2/4$ with equality charateristic of the Hyperbolic geometry.

In may paper "Spectral radius and Amenability in Hilbert Geometries", Houston J. of Math, 35(4), 2009, I proved the inequality.

In the case of C^1 divisible convex set, the answer is yes thanks to M. Crampon results, for the bottom of the spectrum is always less than a quarter of the volume entropy squared.

23.b Is it true that the convex which are amenable are those admitting a polytope in the closure of their orbit by the action of the projective transformations ?

Question 24 : Regarding Volumes of balls

24.a Recently (in a note to appear promptly on the Arxiv) I showed that ther exists a universal constant a_n such that any ball of radius r in any hilbert geometry has a volume bigger that $a_n r^n$. Prove that the lower bound on a_n is the volume of the unit ball of the simplexe, and that equality is only attained in a simplex (*this is some kind of "easy" isoperimetric inequality*).

24.b Does there exists an universal constant a_n such that the assymptotic volume of a polytope in $\mathbb{R}^n$ with k verticies is exactly $k \times a_n$?

24.c Is it true that for all integers n, there exists an affine function f_n such that for a polytopes in $\mathbb{R}^n$ with N verticies, the volume of a ball of radius r is less that $f_n(N)R^n$. This is true in dimension 2 and 3 as I hope to explain during my talk.

24.d With G. Berck and A. Bernig "Volume entropy of Hilbert geometry" Pacific J. of math , 245(2), 2010, we defined the pseudo Gauss curvature of a convex set. The fact that it's square root is in L^1 for any plane convex body allowed us to prove the upper entropy bound conjecture in the plane. Is this function always in L^1 ? A yes to that question implies, by Theorem 3.1 in our paper, the entropy upper bound conjecture and an asymptotic of the ratio of the volume of the ball of radius r by $sh(r)^{n-1}$.

24.e Is there a volume comparison theorem in Hilbert geometry with the Hyperbolic geometry, i.e., for any Ω and $p \in \Omega$, have we (where H stands for the Hyperbolic geometry), for $R > r$,

$$\frac{\mathrm{Vol}_\Omega\big(B_\Omega(p,R)\big)}{\mathrm{Vol}_\Omega\big(B_\Omega(p,r)\big)} \leq \frac{\mathrm{Vol}_H\big(B_H(o,R)\big)}{\mathrm{Vol}_H\big(B_H(o,r)\big)} ?$$

Or in other words, is it true that the function $\mathrm{Vol}_\Omega\big(B_\Omega(p,r)\big)/\mathrm{Vol}_H\big(B_H(o,r)\big)$ is a non-increasing function as $r \to +\infty$.

Remark that it sufficies to prove this statement for a dense subset of the set of convex sets, and that it would also yield the entropy upper bound conjecture.

24.f For what dimension have we equality between the entropy and twice the approximability ?

Question 25 : As seen above for finite metric spaces: what metric spaces can be imbedded into a Hilbert geometry in general, and in a Hilbert geometry of same dimension in particular. For instance is it possible to imbed the three dimensional Heisenberg group in a three dimensional Hilbert geometry (I think no).

Question 26 : Consider in a Hilbert geometry (with C^2 boundary to start with) a point p, and for any $R > 0$ a polytope P_R with minimal verticies which contains the ball of radius $R - 1$ and is contained in the ball of radius $R + 1$, both centred at p. Compute the difference (resp. the ratio) between the hilbert volume of the ball $B(p, r)$ and the hilbert volume of P_R. Study the behaviour as $R \to +\infty$. Does this yield a functional related to the centro projective area (it has to be a projective invariant anyway).

Question 27 : The condition of a Riemannian metric to have some curvature condition can be translated into the fact that this metric belongs to some cone in the space of metric. Can we use the Hilbert metric of that cone to study some transformations on the space wich conseves the cone. I think of the Ricci flow for instance. Would be fun if we could prove that a surface or a 3-manifold admitting a positive curvature metric is a sphere in this way.