

# HILBERT GEOMETRY FOR CONVEX POLYGONAL DOMAINS

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ABSTRACT. We prove in this paper that the Hilbert geometry associated with an open convex polygonal set is Lipschitz equivalent to Euclidean plane.

## 1. INTRODUCTION

A *Hilbert domain* in  $\mathbf{R}^m$  is a metric space  $(\mathcal{C}, d_{\mathcal{C}})$ , where  $\mathcal{C}$  is an *open bounded convex* set in  $\mathbf{R}^m$  and  $d_{\mathcal{C}}$  is the distance function on  $\mathcal{C}$  — called the *Hilbert metric* — defined as follows.

Given two distinct points  $p$  and  $q$  in  $\mathcal{C}$ , let  $a$  and  $b$  be the intersection points of the straight line defined by  $p$  and  $q$  with  $\partial\mathcal{C}$  so that  $p = (1 - s)a + sb$  and  $q = (1 - t)a + tb$  with  $0 < s < t < 1$ . Then

$$d_{\mathcal{C}}(p, q) := \frac{1}{2} \ln[a, p, q, b],$$

where

$$[a, p, q, b] := \frac{1 - s}{s} \times \frac{t}{1 - t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points  $(a, p, q, b)$  (see Figure 1). We complete the definition by setting  $d_{\mathcal{C}}(p, p) := 0$ .

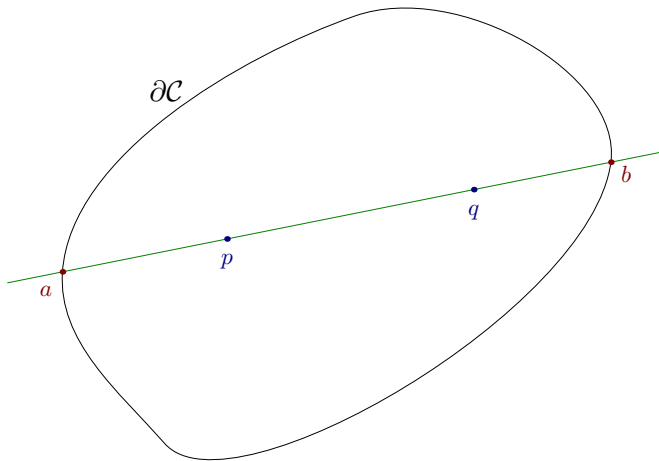


FIGURE 1. The Hilbert metric  $d_{\mathcal{C}}$

The metric space  $(\mathcal{C}, d_{\mathcal{C}})$  thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of  $\mathbf{R}^m$  and in which the affine open segments joining two points of the boundary  $\partial\mathcal{C}$  are geodesics that are isometric to  $(\mathbf{R}, |\cdot|)$ . It is to be mentioned here that in general the affine segment between two points in  $\mathcal{C}$  may *not* be the *unique* geodesic joining these points (for example, if  $\mathcal{C}$  is a square). Nevertheless, this uniqueness holds whenever  $\mathcal{C}$  is *strictly* convex.

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For further information about Hilbert geometry, we refer to [1, 2, 3, 6, 8, 12] and the excellent introduction [11] by Socié-Méthou.

The two fundamental examples of Hilbert domains  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  correspond to the case when  $\mathcal{C}$  is an ellipsoid, which gives the Klein model of  $m$ -dimensional hyperbolic geometry (see for example [11, first chapter]), and the case when  $\bar{\mathcal{C}}$  is a  $m$ -simplex, for which there exists a norm  $\|\cdot\|_{\mathcal{C}}$  on  $\mathbf{R}^m$  such that  $(\mathcal{C}, d_{\mathcal{C}})$  is isometric to the normed vector space  $(\mathbf{R}^m, \|\cdot\|_{\mathcal{C}})$  (see [5, pages 110–113] or [10, pages 22–23]). Therefore, it is natural to study the Hilbert domains  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  for which  $\mathcal{C}$  is close to either an ellipsoid or a  $m$ -simplex.

The first and last authors thus proved in [4] that any Hilbert domain  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^m$  such that the boundary  $\partial\mathcal{C}$  is a  $C^2$  hypersurface with non-vanishing Gaussian curvature is Lipschitz equivalent to  $m$ -dimensional hyperbolic space  $\mathbf{H}^m$ .

On the other hand, Förtsch and Karlsson showed in [7] that a Hilbert domain in  $\mathbf{R}^m$  is isometric to a normed vector space if and only if it is given by a  $m$ -simplex. In addition, Lins established in his PhD thesis [9, Lemma 2.2.5] that the Hilbert geometry associated with an open convex polygonal set in  $\mathbf{R}^2$  can be isometrically embedded in the normed vector space  $(\mathbf{R}^{N^2}, \|\cdot\|_{\infty})$ , where  $N$  is the number of vertices of the polygon.

The aim of this paper is to prove that the Hilbert geometry associated with an open convex *polygonal* set  $\mathcal{P}$  in  $\mathbf{R}^2$  is Lipschitz equivalent to Euclidean plane (Theorem 3.1 in the last section). A straightforward consequence of this result is that *all* the Hilbert *polygonal* domains in  $\mathbf{R}^2$  are Lipschitz equivalent to each other, which is a fact that is far from being obvious at a first glance.

The idea of the proof is to decompose a given open convex  $n$ -sided polygon  $\mathcal{P}$  into  $n$  triangles having one common vertex in  $\mathcal{P}$  and whose opposite edges to that vertex are the sides of  $\mathcal{P}$ , and then to show that each of these triangles is Lipschitz equivalent to the cone it defines with that vertex. This second point is the most technical part of the paper and is based on Proposition 2.2.

**Remark.** It seems that our result might be extended to higher dimensions to prove more generally that any Hilbert domain in  $\mathbf{R}^m$  given by a *polytope* is Lipschitz equivalent to  $m$ -dimensional Euclidean space. Nevertheless, computations in that case appear to be much more difficult since they involve not only the edges of the polytope but also its faces.

## 2. PRELIMINARIES

This section is devoted to some technical properties we will need for the proof of Theorem 3.1 in Section 3. The key results are contained in Proposition 2.1 and Proposition 2.2.

Let us first recall that the distance function  $d_{\mathcal{C}}$  is associated with the Finsler metric  $F_{\mathcal{C}}$  on  $\mathcal{C}$  given, for any  $p \in \mathcal{C}$  and any  $v \in T_p\mathcal{C} = \mathbf{R}^m$  (tangent vector space to  $\mathcal{C}$  at  $p$ ), by

$$F_{\mathcal{C}}(p, v) := \frac{1}{2} \left( \frac{1}{t^-} + \frac{1}{t^+} \right) \quad \text{if } v \neq 0,$$

where  $t^- = t_{\mathcal{C}}^-(p, v)$  and  $t^+ = t_{\mathcal{C}}^+(p, v)$  are the *unique positive* numbers such that  $p - t^-v \in \partial\mathcal{C}$  and  $p + t^+v \in \partial\mathcal{C}$ , and  $F_{\mathcal{C}}(p, 0) := 0$  (see Figure 2).

This means that for every  $p, q \in \mathcal{C}$  and  $v \in T_p\mathcal{C} = \mathbf{R}^m$ , we have  $F_{\mathcal{C}}(p, v) = \frac{d}{dt}\big|_{t=0} d_{\mathcal{C}}(p, p + tv)$  and  $d_{\mathcal{C}}(p, q)$  is the infimum of the length  $\int_0^1 F_{\mathcal{C}}(\sigma(t), \sigma'(t)) dt$  with respect to  $F_{\mathcal{C}}$  when  $\sigma : [0, 1] \rightarrow \mathcal{C}$  ranges over all the  $C^1$  curves joining  $p$  to  $q$ .

**Remark.** For  $p \in \mathcal{C}$  and  $v \in T_p\mathcal{C} = \mathbf{R}^m$  with  $v \neq 0$ , we will define  $p^- = p_{\mathcal{C}}^-(p, v) := p - t_{\mathcal{C}}^-(p, v)v$  and  $p^+ = p_{\mathcal{C}}^+(p, v) := p + t_{\mathcal{C}}^+(p, v)v$ . Then, given any arbitrary norm  $\|\cdot\|$  on  $\mathbf{R}^m$ , we can write

$$F_{\mathcal{C}}(p, v) = \frac{1}{2} \|v\| \left( \frac{1}{\|p - p^-\|} + \frac{1}{\|p - p^+\|} \right).$$

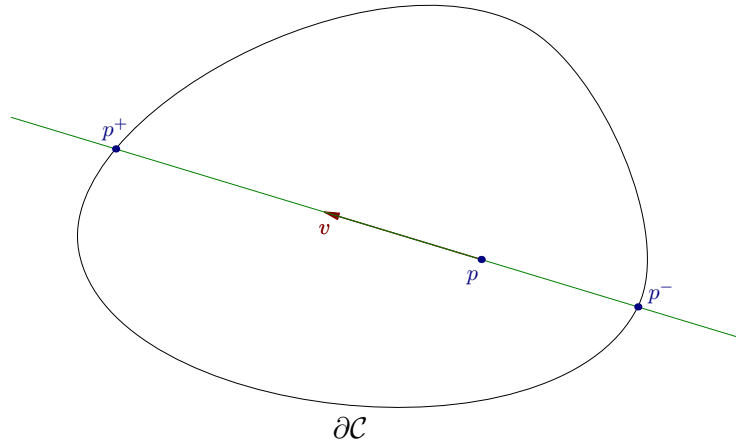


FIGURE 2. The Finsler metric  $F_{\mathcal{C}}$

**Notations.** Let  $\mathcal{S} := ]-1, 1[ \times ]-1, 1[ \subseteq \mathbf{R}^2$  be the standard open square,  $\Delta := \{(x, y) \in \mathbf{R}^2 \mid |y| < x < 1\} \subseteq \mathcal{S}$  the open triangle whose vertices are  $0 = (0, 0)$ ,  $(1, -1)$  and  $(1, 1)$ , and  $\mathcal{Z} := \{(X, Y) \in \mathbf{R}^2 \mid |Y| < X\} \subseteq \mathbf{R}^2$  the open cone associated with  $\Delta$  (see Figure 3).

The canonical basis of  $\mathbf{R}^2$  will be denoted by  $(e_1, e_2)$ .

The usual  $\ell^1$  norm and distance on  $\mathbf{R}^2$  will be denoted respectively by  $\|\cdot\|$  and  $d$ .

**Definition 2.1.** For any pair  $(V_1, V_2)$  of vectors in  $\mathbf{R}^2 \setminus \{0\}$ , the set

$$S(V_1, V_2) := \{sV_1 + tV_2 \mid s \geq 0 \text{ and } t \geq 0\}$$

will be called the *sector* associated with this pair.

**Remark.** The sector  $S(V_1, V_2)$  is the convex hull of the set  $(\mathbf{R}_+V_1) \cup (\mathbf{R}_+V_2)$ .

Let us begin with the following useful lemma:

**Lemma 2.1.** *Given a basis  $(V_1, V_2)$  of  $\mathbf{R}^2$  and a vector  $V \in \mathbf{R}^2$ , we have*

$$V \in S(V_1, V_2) \iff (\det_{(V_1, V_2)}(V_1, V) \geq 0 \text{ and } \det_{(V_1, V_2)}(V, V_2) \geq 0).$$

*Proof.*

The lemma is a mere consequence of the fact that the coordinate system  $(s, t)$  of any vector  $V$  in  $\mathbf{R}^2$  with respect to a basis  $(V_1, V_2)$  of  $\mathbf{R}^2$  is equal to  $(\det_{(V_1, V_2)}(V, V_2), \det_{(V_1, V_2)}(V_1, V))$ .  $\square$

Now, we have

**Proposition 2.1.** *The map  $\Phi : \mathcal{S} \longrightarrow \mathbf{R}^2$  defined by*

$$\Phi(x, y) = (X, Y) := (\operatorname{atanh}(x), \operatorname{atanh}(y))$$

*is a smooth diffeomorphism such that*

- (1)  $\Phi(\Delta) = \mathcal{Z}$ , and
- (2) for all  $m \in \Delta$  and  $V \in T_m \mathcal{S} = \mathbf{R}^2$ ,  $F_{\mathcal{S}}(m, V) \leq \|T_m \Phi \cdot V\| \leq 2F_{\mathcal{S}}(m, V)$ .

Before proving this result, we will need the following (see Figure 3):

**Lemma 2.2.** *Let  $m = (x, y) \in \Delta \subseteq \mathcal{S}$ , and define in  $T_m \mathcal{S} = \mathbf{R}^2$  the vectors*

$$\begin{aligned} V_1 &:= (1, 1) - m = (1 - x, 1 - y), & V_2 &:= m - (1, -1) = (-1 + x, 1 + y), \\ V_3 &:= (-1, 1) - m = (-1 - x, 1 - y) & \text{and} & & V_4 &:= (-1, -1) - m = (-1 - x, -1 - y). \end{aligned}$$

*Then we have the inclusions*

- (1)  $S(V_1, V_2) \subseteq \left\{ V = (\lambda, \mu) \in \mathbf{R}^2 \mid \mu > 0 \text{ and } \frac{|\lambda|}{1-x^2} \leq \frac{\mu}{1-y^2} \right\}$ ,
- (2)  $S(V_2, V_3) \subseteq \{V = (\lambda, \mu) \in \mathbf{R}^2 \mid \lambda < 0 < \mu\}$ ,
- (3)  $S(V_3, V_4) \subseteq \left\{ V = (\lambda, \mu) \in \mathbf{R}^2 \mid \lambda < 0 \text{ and } \frac{|\mu|}{1-y^2} \leq \frac{-\lambda}{1-x^2} \right\}$ , and
- (4)  $S(V_4, -V_1) \subseteq \{V = (\lambda, \mu) \in \mathbf{R}^2 \mid \lambda < 0 \text{ and } \mu < 0\}$ .

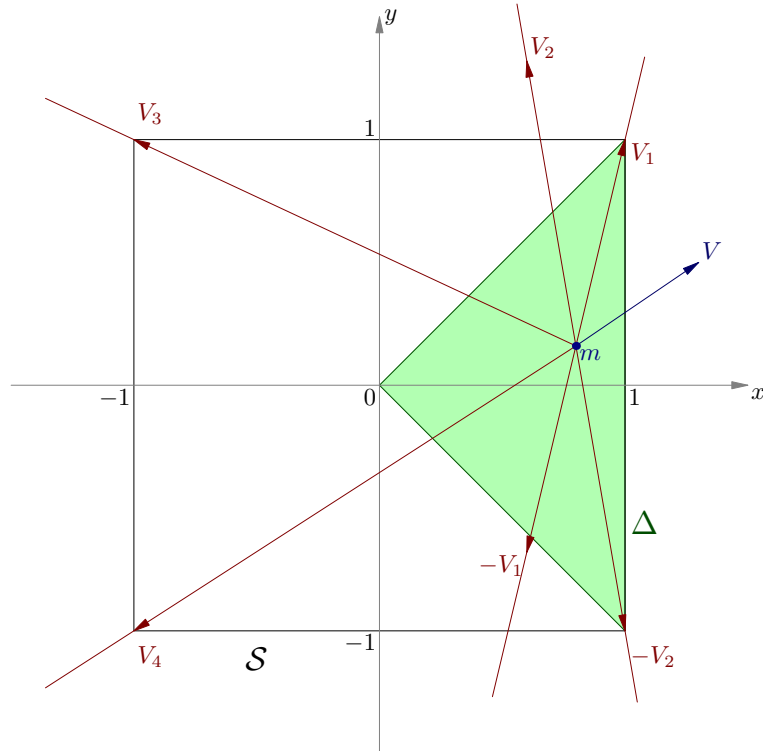


FIGURE 3. The six zones for the vector  $V$

*Proof.*

First of all, we have  $\det_{(e_1, e_2)}(V_1, V_2) = 2(1-x) > 0$ ,  $\det_{(e_1, e_2)}(V_2, V_3) = 2(x+y) > 0$ ,  $\det_{(e_1, e_2)}(V_3, V_4) = 2(1-x) > 0$  and  $\det_{(e_1, e_2)}(V_4, -V_1) = 2(x-y) > 0$  (since  $x > y$ ).

This shows that  $(V_1, V_2)$ ,  $(V_2, V_3)$ ,  $(V_3, V_4)$  and  $(V_4, -V_1)$  are all bases of  $\mathbf{R}^2$  having the same orientation as  $(e_1, e_2)$ .

Then, let  $V = (\lambda, \mu)$  be an arbitrary vector in  $T_m\mathcal{S} = \mathbf{R}^2$ .

• **Point (1):** If  $V \in S(V_1, V_2)$ , then, according to Lemma 2.1, we have

$0 \leq \det_{(e_1, e_2)}(V_1, V) = (1-x)\mu - (1-y)\lambda$  and  $0 \leq \det_{(e_1, e_2)}(V, V_2) = (1+y)\lambda + (1-x)\mu$  since  $(V_1, V_2)$  is a basis of  $\mathbf{R}^2$  having the same orientation as  $(e_1, e_2)$ .

This writes

$$\frac{\lambda}{1-x} \leq \frac{\mu}{1-y} \quad \text{and} \quad \frac{-\mu}{1+y} \leq \frac{\lambda}{1-x},$$

and hence, multiplying both inequalities by  $\frac{1}{1+x} > 0$ , we get

$$(2.1) \quad \frac{\lambda}{1-x^2} \leq \frac{\mu}{(1+x)(1-y)} \quad \text{and} \quad \frac{-\mu}{(1+x)(1+y)} \leq \frac{\lambda}{1-x^2}.$$

On the other hand, writing  $V = sV_1 + tV_2$  with

$$s = \det_{(V_1, V_2)}(V, V_2) \geq 0 \quad \text{and} \quad t = \det_{(V_1, V_2)}(V_1, V) \geq 0,$$

the second coordinate  $\mu$  of  $V$  with respect to the canonical basis  $(e_1, e_2)$  of  $\mathbf{R}^2$  equals

$$\mu = \det_{(e_1, e_2)}(e_1, V) = s \det_{(e_1, e_2)}(e_1, V_1) + t \det_{(e_1, e_2)}(e_1, V_2) = s(1-y) + t(1+y) > 0.$$

This yields  $\frac{\mu}{(1+x)(1-y)} \leq \frac{\mu}{1-y^2}$  since  $0 < 1+y \leq 1+x$ , and hence

$$(2.2) \quad \frac{\lambda}{1-x^2} \leq \frac{\mu}{1-y^2}$$

from the first part of Equation 2.1.

Moreover, we also have  $\frac{-\mu}{1-y^2} \leq \frac{-\mu}{(1+x)(1+y)}$  since  $0 < 1-y \leq 1+x$ . Thus,

$$(2.3) \quad \frac{-\mu}{1-y^2} \leq \frac{\lambda}{1-x^2}$$

from the second part of Equation 2.1.

Finally, summarizing Equations 2.2 and 2.3, we obtain  $\frac{|\lambda|}{1-x^2} \leq \frac{\mu}{1-y^2}$ .

• **Point (2):** If  $V \in S(V_2, V_3)$ , let us write  $V = sV_2 + tV_3$  with

$$s = \det_{(V_2, V_3)}(V, V_3) \geq 0 \quad \text{and} \quad t = \det_{(V_2, V_3)}(V_2, V) \geq 0.$$

Then the first coordinate  $\lambda$  of  $V$  with respect to the canonical basis  $(e_1, e_2)$  of  $\mathbf{R}^2$  equals

$$\lambda = \det_{(e_1, e_2)}(V, e_2) = s \det_{(e_1, e_2)}(V_2, e_2) + t \det_{(e_1, e_2)}(V_3, e_2) = -s(1-x) - t(1+x) < 0.$$

On the other hand, the second coordinate  $\mu$  of  $V$  with respect to  $(e_1, e_2)$  is equal to

$$\mu = \det_{(e_1, e_2)}(e_1, V) = s \det_{(e_1, e_2)}(e_1, V_2) + t \det_{(e_1, e_2)}(e_1, V_3) = s(1+y) + t(1-y) > 0.$$

• **Point (3):** If  $V \in S(V_3, V_4)$ , then, according to Lemma 2.1, we have

$0 \leq \det_{(e_1, e_2)}(V_3, V) = -(1+x)\mu - (1-y)\lambda$  and  $0 \leq \det_{(e_1, e_2)}(V, V_4) = -(1+y)\lambda + (1+x)\mu$

since  $(V_3, V_4)$  is a basis of  $\mathbf{R}^2$  having the same orientation as  $(e_1, e_2)$ .

This writes

$$\frac{\mu}{1-y} \leq \frac{-\lambda}{1+x} \quad \text{and} \quad \frac{\lambda}{1+x} \leq \frac{\mu}{1+y},$$

and hence, multiplying the first inequality by  $\frac{1}{1+y} > 0$  and the second one by  $\frac{1}{1-y} > 0$ , we get

$$(2.4) \quad \frac{\mu}{1-y^2} \leq \frac{-\lambda}{(1+x)(1+y)} \quad \text{and} \quad \frac{\lambda}{(1+x)(1-y)} \leq \frac{\mu}{1-y^2}.$$

On the other hand, writing  $V = sV_3 + tV_4$  with

$$s = \det_{(V_3, V_4)}(V, V_4) \geq 0 \quad \text{and} \quad t = \det_{(V_3, V_4)}(V_3, V) \geq 0,$$

the first coordinate  $\mu$  of  $V$  with respect to the canonical basis  $(e_1, e_2)$  of  $\mathbf{R}^2$  equals

$$\lambda = \det_{(e_1, e_2)}(V, e_2) = s \det_{(e_1, e_2)}(V_3, e_2) + t \det_{(e_1, e_2)}(V_4, e_2) = -s(1+x) - t(1+x) < 0.$$

This yields  $\frac{-\lambda}{(1+x)(1+y)} \leq \frac{-\lambda}{1-x^2}$  since  $0 < 1-x \leq 1+y$ , and hence

$$(2.5) \quad \frac{\mu}{1-y^2} \leq \frac{-\lambda}{1-x^2}$$

from the first part of Equation 2.4.

Moreover, we also have  $\frac{\lambda}{1-x^2} \leq \frac{\lambda}{(1+x)(1-y)}$  since  $0 < 1-x \leq 1-y$ . Thus,

$$(2.6) \quad \frac{\lambda}{1-x^2} \leq \frac{\mu}{1-y^2}$$

from the second part of Equation 2.4.

Finally, summarizing Equations 2.5 and 2.6, we obtain  $\frac{|\mu|}{1-y^2} \leq \frac{-\lambda}{1-x^2}$ .

• **Point (4):** If  $V \in S(V_4, -V_1)$ , let us write  $V = sV_4 - tV_1$  with

$$s = \det_{(V_4, -V_1)}(V, V_4) \geq 0 \quad \text{and} \quad t = \det_{(V_4, -V_1)}(-V_1, V) \geq 0.$$

Then the first coordinate  $\lambda$  of  $V$  with respect to the canonical basis  $(e_1, e_2)$  of  $\mathbf{R}^2$  equals

$$\lambda = \det_{(e_1, e_2)}(V, e_2) = s \det_{(e_1, e_2)}(V_4, e_2) - t \det_{(e_1, e_2)}(V_1, e_2) = -s(1+x) - t(1-x) < 0.$$

On the other hand, the second coordinate  $\mu$  of  $V$  with respect to  $(e_1, e_2)$  is equal to

$$\mu = \det_{(e_1, e_2)}(e_1, V) = s \det_{(e_1, e_2)}(e_1, V_4) - t \det_{(e_1, e_2)}(e_1, V_1) = -s(1+y) - t(1-y) < 0.$$

□

*Proof of Proposition 2.1.*

Only the second point has to be proved since the first one is obvious.

So, fix  $m = (x, y) \in \Delta \subseteq \mathcal{S}$  and  $V = (\lambda, \mu) \in T_m \mathcal{S} = \mathbf{R}^2$  such that  $V \neq 0$ .

A straightforward computation shows that

$$\|T_m \Phi \cdot V\| = \left( \frac{\lambda}{1-x^2}, \frac{\mu}{1-y^2} \right),$$

and thus

$$\|T_m \Phi \cdot V\| = \frac{|\lambda|}{1-x^2} + \frac{|\mu|}{1-y^2}.$$

Now, let us define the vectors  $V_1, V_2, V_3$  and  $V_4$  in  $T_m\mathcal{S} = \mathbf{R}^2$  as in Lemma 2.2.

Since  $\mathbf{R}^2$  is equal to the union of the sectors  $S(V_1, V_2), S(V_2, V_3), S(V_3, V_4), S(V_4, -V_1)$  and their images by the symmetry about the origin 0, and since the Finsler metric  $F_S$  on  $\mathcal{S}$  is reversible, there are four cases to be considered.

• **Case 1:**  $V \in S(V_1, V_2)$ .

The unique positive numbers  $\tau^-$  and  $\tau^+$  such that  $m - \tau^-V \in \partial\mathcal{S}$  and  $m + \tau^+V \in \partial\mathcal{S}$  satisfy  $y - \tau^-\mu = -1$  and  $y + \tau^+\mu = 1$ . So,  $\tau^- = (1 + y)/\mu$  and  $\tau^+ = (1 - y)/\mu$ , and hence

$$F_S(m, V) = \frac{1}{2} \left( \frac{1}{\tau^-} + \frac{1}{\tau^+} \right) = \frac{\mu}{1 - y^2}.$$

But

$$\|T_m\Phi \cdot V\| \geq \frac{|\mu|}{1 - y^2} = \frac{\mu}{1 - y^2}$$

since  $\mu > 0$  by point (1) in Lemma 2.2.

Therefore, we have

$$F_S(m, V) \leq \|T_m\Phi \cdot V\|.$$

On the other hand, point (1) in Lemma 2.2 yields

$$\|T_m\Phi \cdot V\| = \frac{|\lambda|}{1 - x^2} + \frac{\mu}{1 - y^2} \leq \frac{2\mu}{1 - y^2},$$

which shows that

$$\|T_m\Phi \cdot V\| \leq 2F_S(m, V).$$

• **Case 2:**  $V \in S(V_2, V_3)$ .

The unique positive numbers  $\tau^-$  and  $\tau^+$  such that  $m - \tau^-V \in \partial\mathcal{S}$  and  $m + \tau^+V \in \partial\mathcal{S}$  satisfy  $x - \tau^-\lambda = 1$  and  $y + \tau^+\mu = 1$ . So,  $\tau^- = -(1 - x)/\lambda$  and  $\tau^+ = (1 - y)/\mu$ , and hence

$$F_S(m, V) = \frac{1}{2} \left( \frac{-\lambda}{1 - x} + \frac{\mu}{1 - y} \right).$$

But point (2) in Lemma 2.2 implies

$$\|T_m\Phi \cdot V\| = \frac{-\lambda}{1 - x^2} + \frac{\mu}{1 - y^2} = \left( \frac{1}{1 + x} \right) \left( \frac{-\lambda}{1 - x} \right) + \left( \frac{1}{1 + y} \right) \left( \frac{\mu}{1 - y} \right)$$

with  $\frac{-\lambda}{1 - x} > 0$  and  $\frac{\mu}{1 - y} > 0$ .

Therefore, since  $\frac{1}{2} \leq \frac{1}{1 + x} \leq 1$  and  $\frac{1}{2} \leq \frac{1}{1 + y} \leq 1$ , we have

$$F_S(m, V) \leq \|T_m\Phi \cdot V\| \leq 2F_S(m, V).$$

• **Case 3:**  $V \in S(V_3, V_4)$ .

The unique positive numbers  $\tau^-$  and  $\tau^+$  such that  $m - \tau^-V \in \partial\mathcal{S}$  and  $m + \tau^+V \in \partial\mathcal{S}$  satisfy  $x - \tau^-\lambda = 1$  and  $x + \tau^+\lambda = -1$ . So,  $\tau^- = -(1 - x)/\lambda$  and  $\tau^+ = -(1 + x)/\lambda$ , and hence

$$F_S(m, V) = \frac{-\lambda}{1 - x^2}.$$

But

$$\|T_m\Phi \cdot V\| \geq \frac{|\lambda|}{1 - x^2} = \frac{-\lambda}{1 - x^2}$$

since  $\lambda < 0$  by point (3) in Lemma 2.2.

Therefore, we have

$$F_S(m, V) \leq \|T_m \Phi \cdot V\|.$$

On the other hand, point (3) in Lemma 2.2 yields

$$\|T_m \Phi \cdot V\| = \frac{-\lambda}{1-x^2} + \frac{|\mu|}{1-y^2} \leq \frac{-2\lambda}{1-x^2},$$

which shows that

$$\|T_m \Phi \cdot V\| \leq 2F_S(m, V).$$

• **Case 4:**  $V \in S(V_4, -V_1)$ .

The unique positive numbers  $\tau^-$  and  $\tau^+$  such that  $m - \tau^-V \in \partial\mathcal{S}$  and  $m + \tau^+V \in \partial\mathcal{S}$  satisfy  $x - \tau^-\lambda = 1$  and  $y + \tau^+\mu = -1$ . So,  $\tau^- = -(1-x)/\lambda$  and  $\tau^+ = -(1+y)/\mu$ , and hence

$$F_S(m, V) = \frac{1}{2} \left( \frac{-\lambda}{1-x} + \frac{-\mu}{1+y} \right).$$

But point (2) in Lemma 2.2 implies

$$\|T_m \Phi \cdot V\| = \frac{-\lambda}{1-x^2} + \frac{-\mu}{1-y^2} = \left( \frac{1}{1+x} \right) \left( \frac{-\lambda}{1-x} \right) + \left( \frac{1}{1+y} \right) \left( \frac{-\mu}{1-y} \right)$$

with  $\frac{-\lambda}{1-x} > 0$  and  $\frac{-\mu}{1-y} > 0$ .

Therefore, since  $\frac{1}{2} \leq \frac{1}{1+x} \leq 1$  and  $\frac{1}{2} \leq \frac{1}{1+y} \leq 1$ , we have

$$F_S(m, V) \leq \|T_m \Phi \cdot V\| \leq 2F_S(m, V).$$

□

**Remark.** It is to be pointed out that the Lipschitz constants 1 and 2 obtained in Proposition 2.1 are optimal. Indeed, taking  $m = (1/2, 0) \in \Delta$  and  $V = (0, 1) \in S(V_1, V_2) \subseteq T_m\mathcal{S} = \mathbf{R}^2$ , we get  $\|T_m \Phi \cdot V\| = F_S(m, V)$ . On the other hand, we have  $\frac{\|T_m \Phi \cdot V\|}{F_S(m, V)} \rightarrow 2$  when  $m \rightarrow (0, 0)$  and  $V \rightarrow (1, 1)$  with  $m \in \Delta$  and  $V \in S(V_1, V_2) \subseteq T_m\mathcal{S} = \mathbf{R}^2$ .

Given real numbers  $a \in (0, 1)$  and  $c > b \geq 1$ , let  $\mathcal{T} \subseteq \mathbf{R}^2$  be the triangle defined as the open convex hull of the points  $(1, -1)$ ,  $(1, 1)$  and  $(-a, 0)$ , and let  $\mathcal{Q} \subseteq \mathbf{R}^2$  be the quadrilateral defined as the open convex hull of the points  $(1, -1)$ ,  $(1, 1)$ ,  $(-b, c)$  and  $(-b, -c)$  (see Figure 4).

Then we have  $\Delta \subseteq \mathcal{T} \subseteq \mathcal{Q}$  and

**Proposition 2.2.** *There exists a constant  $A = A(a, b, c) \in (0, 1]$  such that*

$$AF_{\mathcal{T}}(m, V) \leq F_{\mathcal{Q}}(m, V) \leq F_{\mathcal{T}}(m, V)$$

for all  $m \in \Delta$  and  $V \in T_m\mathcal{T} = T_m\mathcal{Q} = \mathbf{R}^2$ .

**Remark.** This is the key result of this section, but also the most technical one of the paper. So, the reader may skip it in a first reading without any loss of keeping track of the ideas that lead to the final theorem in Section 3.

Before proving Proposition 2.2, we will need the following simple but very useful fact (see Figure 5):



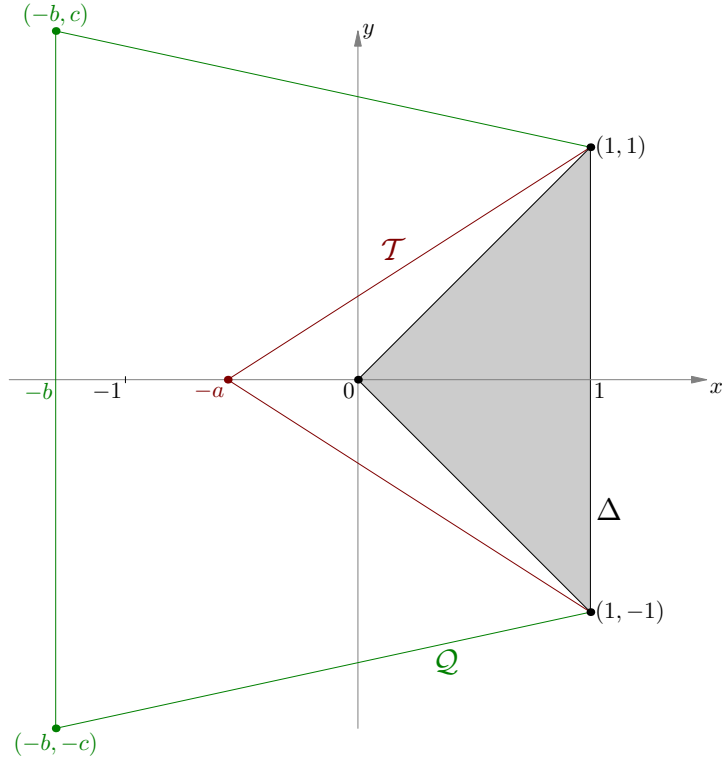


FIGURE 4. The triangle  $\mathcal{T}$  and the quadrilateral  $\mathcal{Q}$

**Lemma 2.3.** Let  $\omega$ ,  $q_1$  and  $q_2$  be non-collinear points in  $\mathbf{R}^2$ , and consider  $p_1 \in ]\omega, q_1[$  and  $p_2 \in ]\omega, q_2[$  such that the lines  $(p_1p_2)$  and  $(q_1q_2)$  intersect in a point  $\omega_0$ .

If  $q_1 \in ]\omega_0, q_2[$ , then we have  $\frac{\overline{\omega q_2}}{\overline{\omega p_2}} > \frac{\overline{\omega q_1}}{\overline{\omega p_1}}$ .

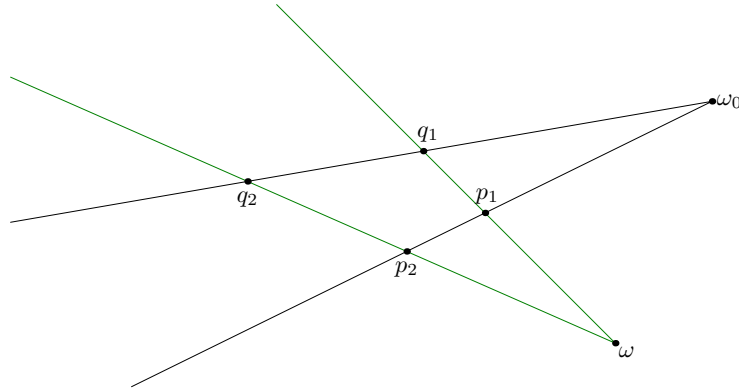


FIGURE 5. Intersecting pairs of half-lines

*Proof.*

Let  $\pi : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$  be the projection of  $\mathbf{R}^2$  onto the line  $(\omega q_2)$  along the direction of  $(\omega_0 p_2)$ .

Since  $\pi$  is affine, it is barycentre-preserving, and hence  $q_1 \in ]\omega_0, q_2[$  necessarily implies  $\pi(q_1) \in ]\pi(\omega_0), \pi(q_2)[ = ]p_2, q_2[$ . So,  $\frac{\overline{\omega q_2}}{\overline{\omega p_2}} > \frac{\overline{\omega \pi(q_1)}}{\overline{\omega p_2}}$ .

But,  $\pi$  being affine with  $\pi(\omega) = \omega$  and  $\pi(p_1) = p_2$ , we also have  $\frac{\overline{\omega \pi(q_1)}}{\overline{\omega p_2}} = \frac{\overline{\omega q_1}}{\overline{\omega p_1}}$ , which proves Lemma 2.3 thanks to the previous inequality.  $\square$

*Proof of Proposition 2.2.*

Since  $\mathcal{T} \subseteq \mathcal{Q}$ , we already have the second inequality. So, the very thing to be proved here is the first inequality.

Recall that  $\|\cdot\|$  and  $d$  denote respectively the usual  $\ell^1$  norm and distance on  $\mathbf{R}^2$ .

Define  $\kappa_0 := \text{diam}_d(\mathcal{Q}) > 0$ , the diameter of  $\mathcal{Q}$  with respect to  $d$ , and let  $\theta_0$  be the intersection point of the line  $\mathbf{R}(1, 1)$  with the line passing through the points  $(1, -1)$  and  $(-b, c)$ ; in other words,  $\theta_0 = (\alpha_0, \alpha_0)$  with  $\alpha_0 := \frac{c-b}{c+b+2} \in (0, 1)$ .

Next consider

$$\Delta^+ := \{(x, y) \in \mathbf{R}^2 \mid 0 \leq y < x < 1\} \subseteq \Delta \quad \text{and} \quad \Sigma := \{(x, y) \in \mathbf{R}^2 \mid \alpha_0 \leq y < x < 1\} \subseteq \Delta^+,$$

fix  $m = (x, y) \in \Sigma$ , and define in  $T_m\mathcal{T} = T_m\mathcal{Q} = \mathbf{R}^2$  the vectors

$$\begin{aligned} V_1 &:= (1, 1) - m = (1 - x, 1 - y), & V_2 &:= m - (1, -1) = (-1 + x, 1 + y), \\ V_3 &:= (-b, c) - m = (-b - x, c - y) & \text{and} & \quad V_4 := (-a, 0) - m = (-a - x, -y). \end{aligned}$$

Then we have

$$\begin{aligned} \det_{(e_1, e_2)}(V_1, V_2) &= \det_{(e_1, e_2)}(-V_2, V_1) = \det_{(e_1, e_2)}(-V_1, -V_2) = 2(1 - x) > 0, \\ \det_{(e_1, e_2)}(V_2, V_3) &= \det_{(e_1, e_2)}(-V_2, -V_3) \\ &= (1 + c)x + (1 + b)y + b - c \geq (2 + b + c)y + b - c \geq 0 \quad (\text{since } x \geq y \geq \alpha_0), \\ \det_{(e_1, e_2)}(V_1, V_3) &= \det_{(e_1, e_2)}(-V_3, V_1) \\ &= (1 - c)x - (1 + b)y + b + c > (b + c)(1 - x) > 0 \quad (\text{since } x > y), \\ \det_{(e_1, e_2)}(V_3, V_4) &= (b - a)y + c(x + a) > 0 \quad (\text{since } b \geq 1 \geq a), \\ \det_{(e_1, e_2)}(V_2, V_4) &= \det_{(e_1, e_2)}(V_4, -V_2) = \det_{(e_1, e_2)}(-V_2, -V_4) = (1 + a)y + x + a > 0, \quad \text{and} \\ \det_{(e_1, e_2)}(V_1, V_4) &= \det_{(e_1, e_2)}(V_4, -V_1) = \det_{(e_1, e_2)}(-V_4, V_1) \\ &= a(1 - y) + x - y > a(1 - y) > 0 \quad (\text{since } x > y). \end{aligned}$$

This shows that  $(V_1, V_3)$ ,  $(V_1, V_4)$ ,  $(V_3, V_4)$ ,  $(V_4, -V_2)$  and  $(-V_2, V_1)$  are bases of  $\mathbf{R}^2$  having the same orientation as  $(e_1, e_2)$  with  $V_2 \in S(V_1, V_3) \cap S(V_1, V_4)$ ,  $V_3 \in S(V_1, V_4)$ ,  $-V_1 \in S(V_4, -V_2)$  and  $-V_3, -V_4 \in S(-V_2, V_1)$ .

Given an arbitrary vector  $V = (\lambda, \mu) \in T_m\mathcal{T} = T_m\mathcal{Q} = \mathbf{R}^2$  such that  $V \neq 0$ , there are now four cases to be dealt with.

- **Case 1:**  $V \in S(V_1, V_2)$  (see Figure 6).

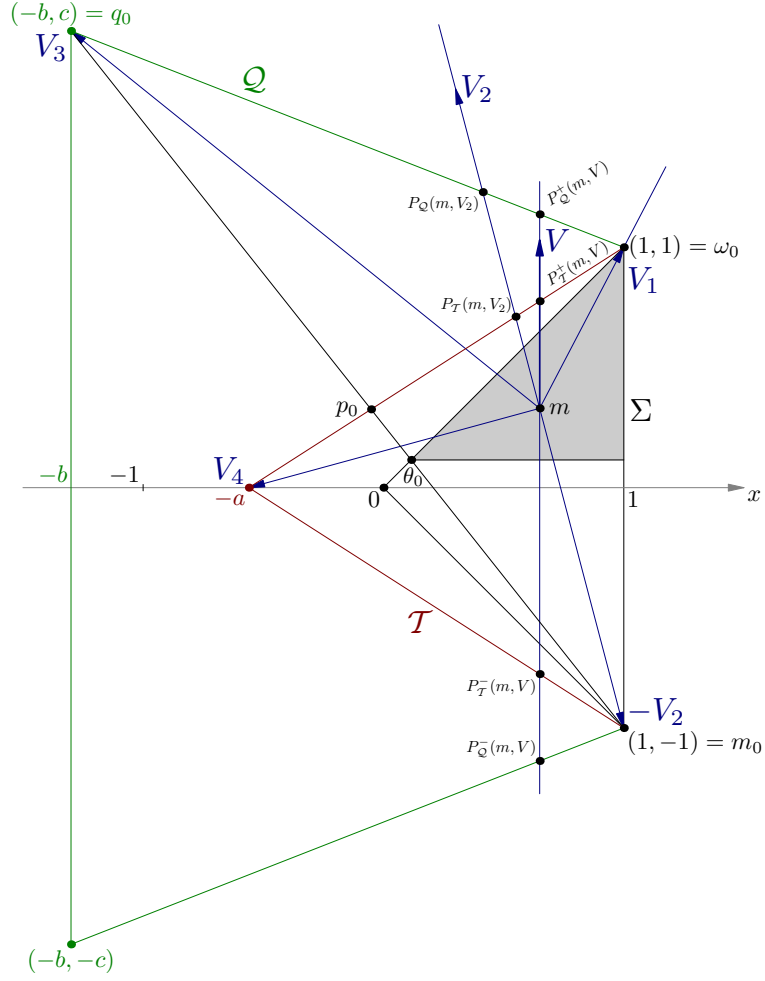
From  $V_2 \in S(V_1, V_3)$ , the half-line  $m + \mathbf{R}_+V_2$  intersects with the segment  $[(1, 1), (-b, c)] \subseteq \partial\mathcal{Q}$ , and hence the same holds for the half-line  $m + \mathbf{R}_+V$  since  $V \in S(V_1, V_2) \subseteq S(V_1, V_3)$ .

Moreover, since  $V_2 \in S(V_1, V_4)$ , we have  $V \in S(V_1, V_2) \subseteq S(V_1, V_4)$ , and this implies that the half-lines  $m + \mathbf{R}_+V_2$  and  $m + \mathbf{R}_+V$  also intersect with the segment  $[(1, 1), (-a, 0)] \subseteq \partial\mathcal{T}$ .

Therefore, if  $V \notin \{V_1, V_2\}$ , Lemma 2.3 with  $\omega := m$ ,  $p_1 := p_{\mathcal{T}}^+(m, V)$ ,  $q_1 := p_{\mathcal{Q}}^+(m, V)$ ,  $p_2 := p_{\mathcal{T}}^+(m, V_2)$ ,  $q_2 := p_{\mathcal{Q}}^+(m, V_2)$  and  $\omega_0 := (1, 1)$  gives

$$(2.7) \quad \frac{t_{\mathcal{Q}}^+(m, V_2)}{t_{\mathcal{T}}^+(m, V_2)} \geq \frac{t_{\mathcal{Q}}^+(m, V)}{t_{\mathcal{T}}^+(m, V)},$$

and this still holds when  $V := V_1$  or  $V := V_2$ .

FIGURE 6. The case when  $V \in S(V_1, V_2)$ 

On the other hand, if  $m_0 = (1, -1)$ ,  $q_0 = (-b, c)$  and  $p_0$  is the intersection point of the line passing through  $(1, 1)$  and  $(-a, 0)$  with the line  $(m_0q_0)$ , Lemma 2.3 with  $\omega = m_0$ ,  $p_1 = p_T^+(m, V_2)$ ,  $q_1 = p_Q^+(m, V_2)$ ,  $p_2 = p_0$ ,  $q_2 = q_0$  and  $\omega_0 = (1, 1)$  yields

$$(2.8) \quad \frac{\overline{m_0q_0}}{\overline{m_0p_0}} \geq \frac{\overline{m_0p_Q^+(m, V_2)}}{\overline{m_0p_T^+(m, V_2)}}.$$

But  $t_Q^+(m, V_2) = \overline{mp_Q^+(m, V_2)} \geq \overline{mp_T^+(m, V_2)} = t_T^+(m, V_2) > 0$  (since  $\mathcal{T} \subseteq \mathcal{Q}$ ) and  $\overline{m_0m} > 0$ , which implies

$$(2.9) \quad \frac{\overline{m_0p_Q^+(m, V_2)}}{\overline{m_0p_T^+(m, V_2)}} \geq 1 \geq \frac{\overline{m_0p_T^+(m, V_2)}}{\overline{m_0p_Q^+(m, V_2)}} = \frac{\overline{m_0m} + t_T^+(m, V_2)}{\overline{m_0m} + t_Q^+(m, V_2)} \geq \frac{t_T^+(m, V_2)}{t_Q^+(m, V_2)}.$$

Then, combining Equations 2.7, 2.8 and 2.9, we get

$$(2.10) \quad \frac{\overline{m_0q_0}}{\overline{m_0p_0}} \geq \frac{t_Q^+(m, V)}{t_T^+(m, V)}.$$

Furthermore, since  $m \in \Sigma$  and  $p_T^-(m, V) \in [(-a, 0), (1, -1)] \subseteq \mathbf{R} \times (-\infty, 0]$  (indeed,  $-V_1 \in S(V_4, -V_2)$  implies  $-V \in S(-V_1, -V_2) \subseteq S(V_4, -V_2)$ ), we have  $\|m - p_T^-(m, V)\| \geq d(m, \mathbf{R} \times \{0\}) \geq \alpha_0$ , and thus

$$t_T^-(m, V) = \frac{\|m - p_T^-(m, V)\|}{\|V\|} \geq \frac{\alpha_0}{\|V\|}.$$

In addition, since  $m, p_{\overline{\mathcal{Q}}}(m, V) \in \overline{\mathcal{Q}}$  and  $\kappa_0 = \text{diam}_d(\overline{\mathcal{Q}})$ , we also have

$$t_{\overline{\mathcal{Q}}}(m, V) = \frac{\|m - p_{\overline{\mathcal{Q}}}(m, V)\|}{\|V\|} \leq \frac{\kappa_0}{\|V\|}.$$

Hence,

$$(2.11) \quad \frac{t_{\overline{\mathcal{Q}}}(m, V)}{t_{\overline{\mathcal{T}}}(m, V)} \leq \frac{\kappa_0}{\alpha_0}.$$

Finally, if  $K_1 = K_1(a, b, c) = \min\{\alpha_0/\kappa_0, \overline{m_0 p_0}/\overline{m_0 q_0}\} \in (0, 1]$ , Equations 2.10 and 2.11 lead to

$$\frac{1}{t_{\overline{\mathcal{Q}}}(m, V)} + \frac{1}{t_{\overline{\mathcal{T}}}(m, V)} \geq K_1 \left( \frac{1}{t_{\overline{\mathcal{T}}}(m, V)} + \frac{1}{t_{\overline{\mathcal{T}}}(m, V)} \right),$$

or equivalently

$$F_{\overline{\mathcal{Q}}}(m, V) \geq K_1 F_{\overline{\mathcal{T}}}(m, V).$$

- Case 2:  $V \in S(V_2, V_3)$  (see Figure 7).

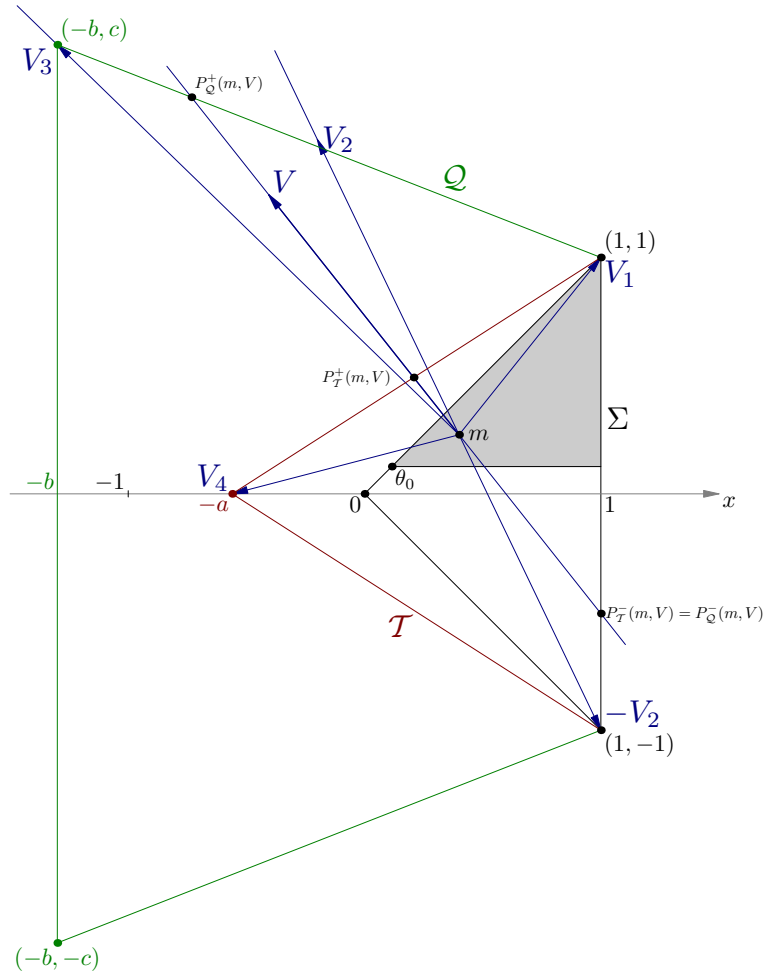


FIGURE 7. The case when  $V \in S(V_2, V_3)$

Since  $V_2 \in S(V_1, V_3)$ , we have  $V \in S(V_2, V_3) \subseteq S(V_1, V_3)$ , and hence the half-line  $m + \mathbf{R}_+ V$  intersects with the segment  $[(1, 1), (-b, c)] \subseteq \partial \overline{\mathcal{Q}}$ .

On the other hand, this half-line also intersects with the segment  $[(1, 1), (-a, 0)] \subseteq \partial \overline{\mathcal{T}}$  since  $V \in S(V_2, V_3) \subseteq S(V_1, V_4)$  (indeed, we have  $V_2, V_3 \in S(V_1, V_4)$ ).

Then, writing that the point  $p_{\mathcal{T}}^+(m, V) = (x + t_{\mathcal{T}}^+(m, V)\lambda, y + t_{\mathcal{T}}^+(m, V)\mu)$  (resp.  $p_{\mathcal{Q}}^+(m, V) = (x + t_{\mathcal{Q}}^+(m, V)\lambda, y + t_{\mathcal{Q}}^+(m, V)\mu)$ ) belongs to the line passing through  $(1, 1)$  and  $(-a, 0)$  (resp.  $(-b, c)$ ) whose equation is  $X - (1 + a)Y + a = 0$  (resp.  $(1 - c)X - (1 + b)Y + (b + c) = 0$ ), we get  $(1 + a)\mu - \lambda > 0$  and  $(c - 1)\lambda + (1 + b)\mu > 0$  together with

$$(2.12) \quad t_{\mathcal{T}}^+(m, V) = \frac{x - (1 + a)y + a}{(1 + a)\mu - \lambda} \quad \text{and} \quad t_{\mathcal{Q}}^+(m, V) = \frac{(1 - c)x - (1 + b)y + (b + c)}{(c - 1)\lambda + (1 + b)\mu}.$$

Next, since  $-V_3 \in S(-V_2, V_1)$ , we have  $-V \in S(-V_2, -V_3) \subseteq S(-V_2, V_1)$ , and hence the half-line  $m - \mathbf{R}_+V$  intersects with the segment  $[(1, -1), (1, 1)] \subseteq \partial\mathcal{T} \cap \partial\mathcal{Q}$ .

The points  $p_{\mathcal{T}}^-(m, V) = (x - t_{\mathcal{T}}^-(m, V)\lambda, y - t_{\mathcal{T}}^-(m, V)\mu)$  and  $p_{\mathcal{Q}}^-(m, V) = (x - t_{\mathcal{Q}}^-(m, V)\lambda, y - t_{\mathcal{Q}}^-(m, V)\mu)$  thus lie on the line passing through  $(1, -1)$  and  $(1, 1)$  whose equation is  $X - 1 = 0$ , which gives  $\lambda < 0$  and

$$(2.13) \quad t_{\mathcal{T}}^-(m, V) = t_{\mathcal{Q}}^-(m, V) = \frac{x - 1}{\lambda}.$$

Now, from Equations 2.12 and 2.13, one obtains

$$2F_{\mathcal{Q}}(m, V) = \frac{1}{t_{\mathcal{Q}}^-(m, V)} + \frac{1}{t_{\mathcal{Q}}^+(m, V)} = \frac{1 + b}{x - 1} \times \frac{\lambda(1 - y) - \mu(1 - x)}{(1 - c)x - (1 + b)y + (b + c)}$$

together with

$$2F_{\mathcal{T}}(m, V) = \frac{1}{t_{\mathcal{T}}^-(m, V)} + \frac{1}{t_{\mathcal{T}}^+(m, V)} = \frac{1 + a}{x - 1} \times \frac{\lambda(1 - y) - \mu(1 - x)}{x - (1 + a)y + a},$$

and hence

$$(2.14) \quad \frac{F_{\mathcal{Q}}(m, V)}{F_{\mathcal{T}}(m, V)} = \frac{1 + b}{1 + a} \times \frac{x - (1 + a)y + a}{(1 - c)x - (1 + b)y + (b + c)}.$$

As  $y \leq x < 1$ , we have both  $x - (1 + a)y + a \geq a(1 - y) > 0$  and  $0 < c(1 - x) + b(1 - y) + (x - y) = (1 - c)x - (1 + b)y + (b + c) \leq (b + c)(1 - y)$  (indeed,  $(1 - c)x \leq (1 - c)y$  since  $c > 1$ ), from which Equation 2.14 finally yields

$$F_{\mathcal{Q}}(m, V) \geq K_2 F_{\mathcal{T}}(m, V),$$

where  $K_2 = K_2(a, b, c) = \frac{a(1 + b)}{(1 + a)(b + c)} \in (0, 1]$ .

• **Case 3:**  $V \in S(V_3, V_4)$  (see Figure 8).

Since  $-V_3, -V_4 \in S(-V_2, V_1)$ , we have  $-V \in S(-V_3, -V_4) \subseteq S(-V_2, V_1)$ , and hence the half-line  $m - \mathbf{R}_+V$  intersects with the segment  $[(1, -1), (1, 1)] \subseteq \partial\mathcal{T} \cap \partial\mathcal{Q}$ .

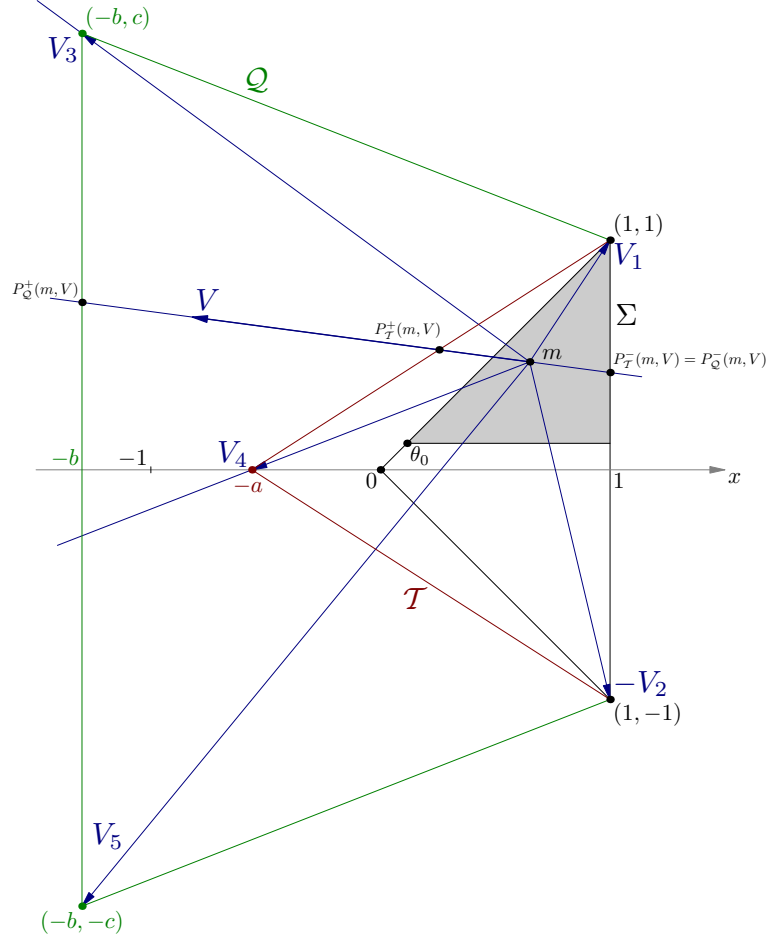
So, we get again  $\lambda < 0$  and

$$(2.15) \quad t_{\mathcal{T}}^-(m, V) = t_{\mathcal{Q}}^-(m, V) = \frac{x - 1}{\lambda}.$$

On the other hand, let  $V_5 = (-b, -c) - m = (-b - x, -c - y)$ .

We have  $\det_{(e_1, e_2)}(V_3, V_4) \geq 0$ ,  $\det_{(e_1, e_2)}(V_4, V_5) = cx + (a - b)y + ac \geq (a + c - b)y + ac \geq 0$  (since  $x \geq y$  and  $c \geq b$ ) and  $\det_{(e_1, e_2)}(V_3, V_5) = 2c(b + x) > 0$ , which shows that  $(V_3, V_5)$  is a basis of  $\mathbf{R}^2$  having the same orientation as  $(e_1, e_2)$  with  $V_4 \in S(V_3, V_5)$ . Therefore,  $V \in S(V_3, V_4) \subseteq S(V_3, V_5)$ , and hence the half-line  $m + \mathbf{R}_+V$  intersects with the segment  $[(-b, c), (-b, -c)] \subseteq \partial\mathcal{Q}$ .

Furthermore, this half-line also intersects with the segment  $[(1, 1), (-a, 0)] \subseteq \partial\mathcal{T}$  since  $V \in S(V_3, V_4) \subseteq S(V_1, V_4)$  (indeed, we have  $V_3 \in S(V_1, V_4)$ ).

FIGURE 8. The case when  $V \in S(V_3, V_4)$ 

Then, writing that the point  $p_T^+(m, V)$  (resp.  $p_Q^+(m, V)$ ) belongs to the line passing through  $(1, 1)$  and  $(-a, 0)$  (resp.  $(-b, c)$  and  $(-b, -c)$ ) whose equation is  $X - (1+a)Y + a = 0$  (resp.  $X + b = 0$ ), we compute  $(1+a)\mu - \lambda > 0$  together with

$$(2.16) \quad t_T^+(m, V) = \frac{x - (1+a)y + a}{(1+a)\mu - \lambda} \quad \text{and} \quad t_Q^+(m, V) = \frac{x + b}{-\lambda}.$$

Equations 2.15 and 2.16 then yield  $\lambda(1-y) - \mu(1-x) < 0$  and

$$(2.17) \quad \frac{F_Q(m, V)}{F_T(m, V)} = \frac{1+b}{1+a} \times \frac{x - (1+a)y + a}{x+b} \times \frac{\lambda}{\lambda(1-y) - \mu(1-x)}.$$

Since  $V \in S(V_3, V_4)$  and  $(V_3, V_4)$  is a basis of  $\mathbf{R}^2$  with the same orientation as  $(e_1, e_2)$ , we have  $\det_{(e_1, e_2)}(V_3, V) = (\lambda y - \mu x) - (c\lambda + b\mu) \geq 0$  and  $\det_{(e_1, e_2)}(V, V_4) = (\lambda y - \mu x) - a\mu \leq 0$ , and thus  $-(b-a)\mu \geq c\lambda$ .

But  $y \leq x < 1$ ,  $b > a$  and  $\lambda < 0$  then imply

$$\begin{aligned} 0 > (b-a)(\lambda(1-y) - \mu(1-x)) &= (b-a)\lambda(1-y) - (b-a)\mu(1-x) \\ &\geq (b-a)\lambda(1-y) + c\lambda(1-x) \\ &\geq (b-a)\lambda(1-y) + c\lambda(1-y) = \lambda(b-a+c)(1-y), \end{aligned}$$

and hence

$$\frac{\lambda}{\lambda(1-y) - \mu(1-x)} = \frac{(b-a)\lambda}{(b-a)(\lambda(1-y) - \mu(1-x))} \geq \frac{b-a}{(b-a+c)(1-y)} > 0.$$

Finally, using  $x - (1 + a)y + a \geq a(1 - y) > 0$  together with  $0 < x + b \leq 1 + b$ , Equation 2.17 gives

$$F_{\mathcal{Q}}(m, V) \geq K_3 F_{\mathcal{T}}(m, V),$$

where  $K_3 = K_3(a, b, c) := \frac{a(b - a)}{(1 + a)(b - a + c)} \in (0, 1]$ .

- **Case 4:**  $V \in S(V_4, -V_1)$  (see Figure 9).

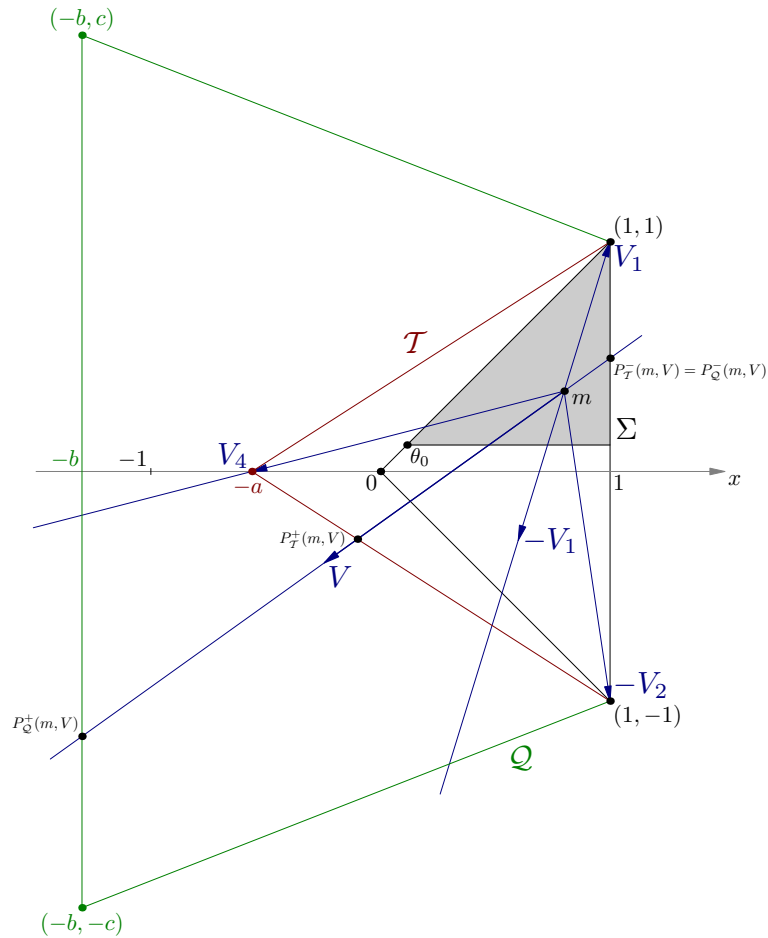


FIGURE 9. The case when  $V \in S(V_4, -V_1)$

Since  $-V_4 \in S(-V_2, V_1)$ , we have  $-V \in S(-V_4, V_1) \subseteq S(-V_2, V_1)$ , and hence the half-line  $m - \mathbf{R}_+ V$  intersects with the segment  $[(1, -1), (1, 1)] \subseteq \partial \mathcal{T} \cap \partial \mathcal{Q}$ .

So,  $t_{\mathcal{T}}^-(m, V) = t_{\mathcal{Q}}^-(m, V)$ , which means that

$$(2.18) \quad \frac{t_{\mathcal{Q}}^-(m, V)}{t_{\mathcal{T}}^-(m, V)} = 1.$$

On the other hand, since  $m \in \Sigma$  and  $p_{\mathcal{T}}^+(m, V) \in [(-a, 0), (1, -1)] \subseteq \mathbf{R} \times (-\infty, 0]$  (indeed,  $-V_1 \in S(V_4, -V_2)$  implies  $V \in S(V_4, -V_1) \subseteq S(V_4, -V_2)$ ), we have  $\|m - p_{\mathcal{T}}^+(m, V)\| \geq d(m, \mathbf{R} \times \{0\}) \geq \alpha_0$ , and thus

$$t_{\mathcal{T}}^+(m, V) = \frac{\|m - p_{\mathcal{T}}^+(m, V)\|}{\|V\|} \geq \frac{\alpha_0}{\|V\|}.$$

In addition, since  $m, p_{\mathcal{Q}}^+(m, V) \in \overline{\mathcal{Q}}$  and  $\kappa_0 = \text{diam}_d(\overline{\mathcal{Q}})$ , we also have

$$t_{\mathcal{Q}}^+(m, V) = \frac{\|m - p_{\mathcal{Q}}^+(m, V)\|}{\|V\|} \leq \frac{\kappa_0}{\|V\|}.$$

Hence,

$$(2.19) \quad \frac{t_{\mathcal{Q}}^+(m, V)}{t_{\mathcal{T}}^+(m, V)} \leq \frac{\kappa_0}{\alpha_0}.$$

Finally, if  $K_4 = K_4(a, b, c) = \min\{1, \alpha_0/\kappa_0\} \in (0, 1]$ , Equations 2.18 and 2.19 lead to

$$F_{\mathcal{Q}}(m, V) \geq K_4 F_{\mathcal{T}}(m, V).$$

At this stage of the proof, defining  $K = K(a, b, c) = \min\{K_1, K_2, K_3, K_4\} \in (0, 1]$  and summing up the results obtained in the four cases discussed above, we can write

$$(2.20) \quad F_{\mathcal{Q}}(m, V) \geq K F_{\mathcal{T}}(m, V)$$

for all  $m \in \Sigma$  and  $V \in T_m \mathcal{T} = T_m \mathcal{Q} = \mathbf{R}^2$ .

Now, the only thing to be done is to establish a similar inequality as in Equation 2.20 for  $m \in \Delta^+ \setminus \Sigma$ , from which we will get Proposition 2.2 since both  $\mathcal{T}$  and  $\mathcal{Q}$  are preserved by the reflection about the  $x$ -axis.

So, let  $\delta_0 = d(\Delta^+ \setminus \Sigma, [(1, 1), (-a, 0)]) > 0$  and consider a point  $m \in \Delta^+ \setminus \Sigma$  together with a vector  $V = (\lambda, \mu) \in \mathbf{R}^2$  such that  $V \neq 0$ .

First of all, since  $m, p_{\mathcal{Q}}^+(m, V) \in \overline{\mathcal{Q}}$  and  $\kappa_0 = \text{diam}_d(\overline{\mathcal{Q}})$ , we have

$$t_{\mathcal{Q}}^+(m, V) = \frac{\|m - p_{\mathcal{Q}}^+(m, V)\|}{\|V\|} \leq \frac{\kappa_0}{\|V\|}.$$

Next, the Finsler metrics  $F_{\mathcal{T}}$  and  $F_{\mathcal{Q}}$  being reversible, we can assume that  $\lambda \leq 0$ , and hence  $p_{\mathcal{T}}^+(m, V) \in \partial \mathcal{T} \setminus (\{1\} \times [-1, 1])$ .

This implies that  $\|m - p_{\mathcal{T}}^+(m, V)\| \geq \delta_0$  and gives

$$t_{\mathcal{T}}^+(m, V) = \frac{\|m - p_{\mathcal{T}}^+(m, V)\|}{\|V\|} \geq \frac{\delta_0}{\|V\|}.$$

Therefore,

$$(2.21) \quad \frac{t_{\mathcal{Q}}^+(m, V)}{t_{\mathcal{T}}^+(m, V)} \leq \frac{\kappa_0}{\delta_0}.$$

On the other hand, as regards  $t_{\mathcal{T}}^-(m, V)$  and  $t_{\mathcal{Q}}^-(m, V)$ , we have two cases to look at.

- **First case:**  $p_{\mathcal{T}}^-(m, V) \in \{1\} \times [-1, 1] \subseteq \partial \mathcal{T}$ .

In that case, we also have  $p_{\mathcal{Q}}^-(m, V) \in \{1\} \times [-1, 1] \subseteq \partial \mathcal{Q}$ , and hence  $t_{\mathcal{T}}^-(m, V) = t_{\mathcal{Q}}^-(m, V)$ , or equivalently

$$(2.22) \quad \frac{t_{\mathcal{Q}}^-(m, V)}{t_{\mathcal{T}}^-(m, V)} = 1.$$

So, if  $K_5 = K_5(a, b, c) = \min\{1, \delta_0/\kappa_0\} \in (0, 1]$ , Equations 2.21 and 2.22 yield

$$(2.23) \quad F_{\mathcal{Q}}(m, V) \geq K_5 F_{\mathcal{T}}(m, V).$$

- **Second case:**  $p_{\mathcal{T}}^-(m, V) \in \partial \mathcal{T} \setminus (\{1\} \times [-1, 1])$ .



Then we have

$$t_{\mathcal{T}}^-(m, V) = \frac{\|m - p_{\mathcal{T}}^-(m, V)\|}{\|V\|} \geq \frac{\delta_0}{\|V\|}.$$

But

$$t_{\mathcal{Q}}^-(m, V) = \frac{\|m - p_{\mathcal{Q}}^-(m, V)\|}{\|V\|} \leq \frac{\kappa_0}{\|V\|}$$

since  $m, p_{\mathcal{Q}}^+(m, V) \in \overline{\mathcal{Q}}$  and  $\kappa_0 = \text{diam}_d(\overline{\mathcal{Q}})$ .

Therefore,

$$(2.24) \quad \frac{t_{\mathcal{Q}}^-(m, V)}{t_{\mathcal{T}}^-(m, V)} \leq \frac{\kappa_0}{\delta_0}.$$

If  $K_6 = K_6(a, b, c) := \delta_0/\kappa_0 \in (0, 1]$ , Equations 2.21 and 2.24 thus lead to

$$(2.25) \quad F_{\mathcal{Q}}(m, V) \geq K_6 F_{\mathcal{T}}(m, V).$$

Conclusion: combining Equations 2.20, 2.23 and 2.25, and defining

$$A = A(a, b, c) := \min\{K, K_5, K_6\} \in (0, 1],$$

we have finally obtained that

$$F_{\mathcal{Q}}(m, V) \geq A F_{\mathcal{T}}(m, V)$$

for all  $m \in \Delta^+$  and  $V \in T_m \mathcal{T} = T_m \mathcal{Q} = \mathbf{R}^2$ , which ends the proof of Proposition 2.2.  $\square$

From Proposition 2.2, we can then deduce

**Proposition 2.3.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be open bounded convex sets in  $\mathbf{R}^2$  such that*

- (1) *the segment  $\{1\} \times [-1, 1]$  is included in both boundaries  $\partial \mathcal{C}_1$  and  $\partial \mathcal{C}_2$ ,*
- (2)  *$(1, -1)$  and  $(1, 1)$  are corner points of  $\overline{\mathcal{C}}_1$  and  $\overline{\mathcal{C}}_2$ ,*
- (3) *the origin  $0$  lies in  $\mathcal{C}_1 \cap \mathcal{C}_2$ , and*
- (4)  *$\Delta \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ .*

*Then there exists a constant  $B = B(\mathcal{C}_1, \mathcal{C}_2) \geq 1$  that satisfies*

$$\frac{1}{B} F_{\mathcal{C}_1}(m, V) \leq F_{\mathcal{C}_2}(m, V) \leq B F_{\mathcal{C}_1}(m, V)$$

*for all  $m \in \Delta$  and  $V \in T_m \mathcal{C}_1 = T_m \mathcal{C}_2 = \mathbf{R}^2$ .*

*Proof.*

Since  $\mathcal{C}_1 \cap \mathcal{C}_2$  is an open set in  $\mathbf{R}^2$  that contains the origin  $0$  by point (3), its intersection  $I$  with  $\mathbf{R} \times \{0\}$  is an open set in  $\mathbf{R} \times \{0\}$  which also contains  $0$ , and hence there exists a number  $a \in (0, 1)$  such that  $[-a, a] \times \{0\} \subseteq I$ . This implies that  $(-a, 0) \in I \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ , and therefore  $\mathcal{T} \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ , where  $\mathcal{T} \subseteq \mathbf{R}^2$  is the triangle defined as the open convex hull of the points  $(1, -1)$ ,  $(1, 1)$  and  $(-a, 0)$  (indeed,  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are convex sets in  $\mathbf{R}^2$  whose boundaries contain  $(1, -1)$  and  $(1, 1)$  by point (1)).

On the other hand, since the sets  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are bounded, there exists a number  $b \geq 1$  such that they are *both* inside  $[-b, b] \times [-b, b]$ . So,  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are included in the open half-plane of  $\mathbf{R}^2$  whose boundary is the line  $\{-b\} \times \mathbf{R}$  and which contains the origin  $0$ .

Next, the convexity of  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ) together with points (1) and (3) show that  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ) lies inside the open half-plane of  $\mathbf{R}^2$  whose boundary is the line  $\mathbf{R}((1, -1) - (1, 1)) = \{1\} \times \mathbf{R}$  and which contains the origin  $0$ .

Moreover, point (2) implies that  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ) has support lines  $\mathcal{L}_1^-$  (resp.  $\mathcal{L}_2^-$ ) and  $\mathcal{L}_1^+$  (resp.  $\mathcal{L}_2^+$ ) at respectively  $(1, -1)$  and  $(1, 1)$  which are not equal to the line  $\{1\} \times \mathbf{R}$ . Therefore,  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ )

lies inside the open half-planes of  $\mathbf{R}^2$  which contain the origin 0 and whose boundaries are the lines  $\mathcal{L}_1^-$  (resp.  $\mathcal{L}_2^-$ ) and  $\mathcal{L}_1^+$  (resp.  $\mathcal{L}_2^+$ ).

So, if we denote by  $c_1$  (resp.  $c_2$ ) the maximum of the absolute values of the second coordinates of the intersection points of the lines  $\mathcal{L}_1^-$  (resp.  $\mathcal{L}_2^-$ ) and  $\mathcal{L}_1^+$  (resp.  $\mathcal{L}_2^+$ ) with the line  $\{-b\} \times \mathbf{R}$ , then  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are included in the open half-planes of  $\mathbf{R}^2$  which contain the origin 0 and whose boundaries are the lines  $\mathbf{R}((1, -1) - (-b, -c))$  and  $\mathbf{R}((-b, c) - (1, 1))$ , where  $c := \max\{c_1, c_2\} + b + 1 > b$ .

Conclusion: we have  $\mathcal{C}_1 \subseteq \mathcal{Q}$  and  $\mathcal{C}_2 \subseteq \mathcal{Q}$ , where  $\mathcal{Q} \subseteq \mathbf{R}^2$  is the quadrilateral defined as the open convex hull of the points  $(1, -1)$ ,  $(1, 1)$ ,  $(-b, c)$  and  $(-b, -c)$ .

Now, for all  $m \in \Delta$  and  $V \in T_m \mathcal{T} = T_m \mathcal{C}_1 = T_m \mathcal{C}_2 = T_m \mathcal{Q} = \mathbf{R}^2$ , we can write

$$\begin{aligned} AF_{\mathcal{C}_1}(m, V) &\leq AF_{\mathcal{T}}(m, V) \quad (\text{since } \mathcal{T} \subseteq \mathcal{C}_1) \\ &\leq F_{\mathcal{Q}}(m, V) \quad (\text{by the first inequality in Proposition 2.2}) \\ &\leq F_{\mathcal{C}_2}(m, V) \quad (\text{since } \mathcal{C}_2 \subseteq \mathcal{Q}) \\ &\leq F_{\mathcal{T}}(m, V) \quad (\text{since } \mathcal{T} \subseteq \mathcal{C}_2) \\ &\leq \frac{1}{A} F_{\mathcal{Q}}(m, V) \quad (\text{by the first inequality in Proposition 2.2}) \\ &\leq \frac{1}{A} F_{\mathcal{C}_1}(m, V) \quad (\text{since } \mathcal{C}_1 \subseteq \mathcal{Q}), \end{aligned}$$

which proves Proposition 2.3 with  $B := 1/A \geq 1$ . □

### 3. LIPSCHITZ EQUIVALENCE TO EUCLIDEAN PLANE

In this section, we build a homeomorphism from an open convex polygonal set to Euclidean plane, and prove that it is bi-Lipschitz with respect to the Hilbert metric of the polygonal set and the Euclidean distance of the plane. This is the statement of Theorem 3.1.

So, let  $\mathcal{P}$  be an open convex polygonal set in  $\mathbf{R}^2$  that contains the origin 0.

Let  $v_1, \dots, v_n$  be the vertices of  $\mathcal{P}$  (*i.e.*, the corner points of the convex set  $\mathcal{P}$ ) that we assume to be cyclically ordered in  $\partial \mathcal{P}$  (notice we have  $n \geq 3$ ).

Define  $v_0 := v_n$  and  $v_{n+1} := v_1$ .

Let  $f : \mathcal{P} \rightarrow \mathbf{R}^2$  be the map defined as follows.

For each  $k \in \{1, \dots, n\}$ , let  $\Delta_k := \{sv_k + tv_{k+1} \mid s \geq 0, t \geq 0 \text{ and } s+t < 1\} \subseteq \mathcal{P}$ , and consider the unique linear transformation  $L_k$  of  $\mathbf{R}^2$  such that  $L_k(v_k) := (1, -1)$  and  $L_k(v_{k+1}) := (1, 1)$ .

Then, given any  $p \in \Delta_k$ , we define

$$f(p) := L_k^{-1}(\Phi(L_k(p))),$$

where  $\Phi : \mathcal{S} \rightarrow \mathbf{R}^2$  is the map considered in Proposition 2.1.

In other words,  $f$  makes the following diagram commute (see Figure 10):

$$\begin{array}{ccc} \Delta_k & \xrightarrow{f} & S(v_k, v_{k+1}) \\ L_k \downarrow & & \downarrow L_k \\ \overline{\Delta} \setminus (\{1\} \times [-1, 1]) & \xrightarrow{\Phi} & \overline{Z} \end{array} \quad ,$$

where we recall that  $\mathcal{S} := ]-1, 1[ \times ]-1, 1[ \subseteq \mathbf{R}^2$ ,  $\Delta := \{(x, y) \in \mathbf{R}^2 \mid |y| < x < 1\} \subseteq \mathcal{S}$  and  $\mathcal{Z} := \{(X, Y) \in \mathbf{R}^2 \mid |Y| < X\} \subseteq \mathbf{R}^2$ .

This makes sense since  $\bigcup_{i=1}^n \Delta_i = \mathcal{P}$ ,  $L_k(\Delta_k) = \overline{\Delta} \setminus (\{1\} \times [-1, 1]) \subseteq \mathcal{S}$  and for all  $p \in [0, 1]v_k = \Delta_{k-1} \cap \Delta_k$ , we have  $L_{k-1}^{-1}(\Phi(L_{k-1}(p))) = L_k^{-1}(\Phi(L_k(p))) \in \mathbf{R}v_k$ .

With this definition and keeping in mind that  $\|\cdot\|$  and  $d$  denote respectively the usual  $\ell^1$  norm and distance on  $\mathbf{R}^2$ . we get

**Theorem 3.1.** *The map  $f$  satisfies the following properties:*

(1)  *$f$  is a homeomorphism.*

(2) *If  $\mathcal{U}$  and  $\mathcal{V}$  are the open sets in  $\mathbf{R}^2$  defined by*

$$\mathcal{U} := \mathcal{P} \setminus \left( \bigcup_{k=1}^n [0, 1]v_k \right) \quad \text{and} \quad \mathcal{V} := \mathbf{R}^2 \setminus \left( \bigcup_{k=1}^n \mathbf{R}v_k \right),$$

*then  $f(\mathcal{U}) = \mathcal{V}$  and  $f$  induces a smooth diffeomorphism from  $\mathcal{U}$  onto  $\mathcal{V}$ .*

(3) *There exists a constant  $C \geq 1$  such that*

$$\frac{1}{C} d_{\mathcal{P}}(p, q) \leq d(f(p), f(q)) \leq C d_{\mathcal{P}}(p, q)$$

*for all  $p, q \in \mathcal{P}$ .*

Before proving this theorem, let us establish the following:

**Lemma 3.1.** *Given any real number  $\alpha > 0$ , there is a constant  $M = M(\alpha) \geq 1$  such that*

$$\frac{1}{M} \ln\left(\frac{1-s}{1-t} \times \frac{1+\alpha t}{1+\alpha s}\right) \leq \ln\left(\frac{1-s}{1-t} \times \frac{1+t}{1+s}\right) \leq M \ln\left(\frac{1-s}{1-t} \times \frac{1+\alpha t}{1+\alpha s}\right)$$

*for all  $0 \leq s < t < 1$ .*

*Proof.*

Consider  $D := \{(s, t) \in \mathbf{R}^2 \mid 0 \leq s < t \leq 1\}$  and let  $\varphi : D \rightarrow \mathbf{R}$  be the function defined by

$$\varphi(s, t) := \ln\left(\frac{1+t}{1+s}\right) / \ln\left(\frac{1+\alpha t}{1+\alpha s}\right).$$

Given  $\lambda \in [0, 1]$ , we have for all  $(s, t) \in D$ ,

$$\varphi(s, t) \sim \left(\frac{1+t}{1+s} - 1\right) / \left(\frac{1+\alpha t}{1+\alpha s} - 1\right) = \frac{1+\alpha s}{\alpha(1+s)} \rightarrow \frac{1+\alpha \lambda}{\alpha(1+\lambda)}$$

as  $(s, t) \rightarrow (\lambda, \lambda)$ .

Hence, by continuity of  $\varphi$ , the function  $\hat{\varphi} : \overline{D} \rightarrow \mathbf{R}$  defined by

$$\hat{\varphi}(s, t) := \varphi(s, t) \quad \text{if } (s, t) \in D \quad \text{and} \quad \hat{\varphi}(s, t) := \frac{1+\alpha s}{\alpha(1+s)} \quad \text{if } s = t$$

is continuous.

Then, compactness of  $\overline{D}$  implies that  $\hat{\varphi}$  has a minimum and a maximum. But these latter are positive since one can easily check that  $\hat{\varphi}(s, t) > 0$  for all  $(s, t) \in \overline{D}$ , and this implies that there is a constant  $M \geq 1$  such that

$$(3.1) \quad \frac{1}{M} \ln\left(\frac{1+\alpha t}{1+\alpha s}\right) \leq \ln\left(\frac{1+t}{1+s}\right) \leq M \ln\left(\frac{1+\alpha t}{1+\alpha s}\right)$$

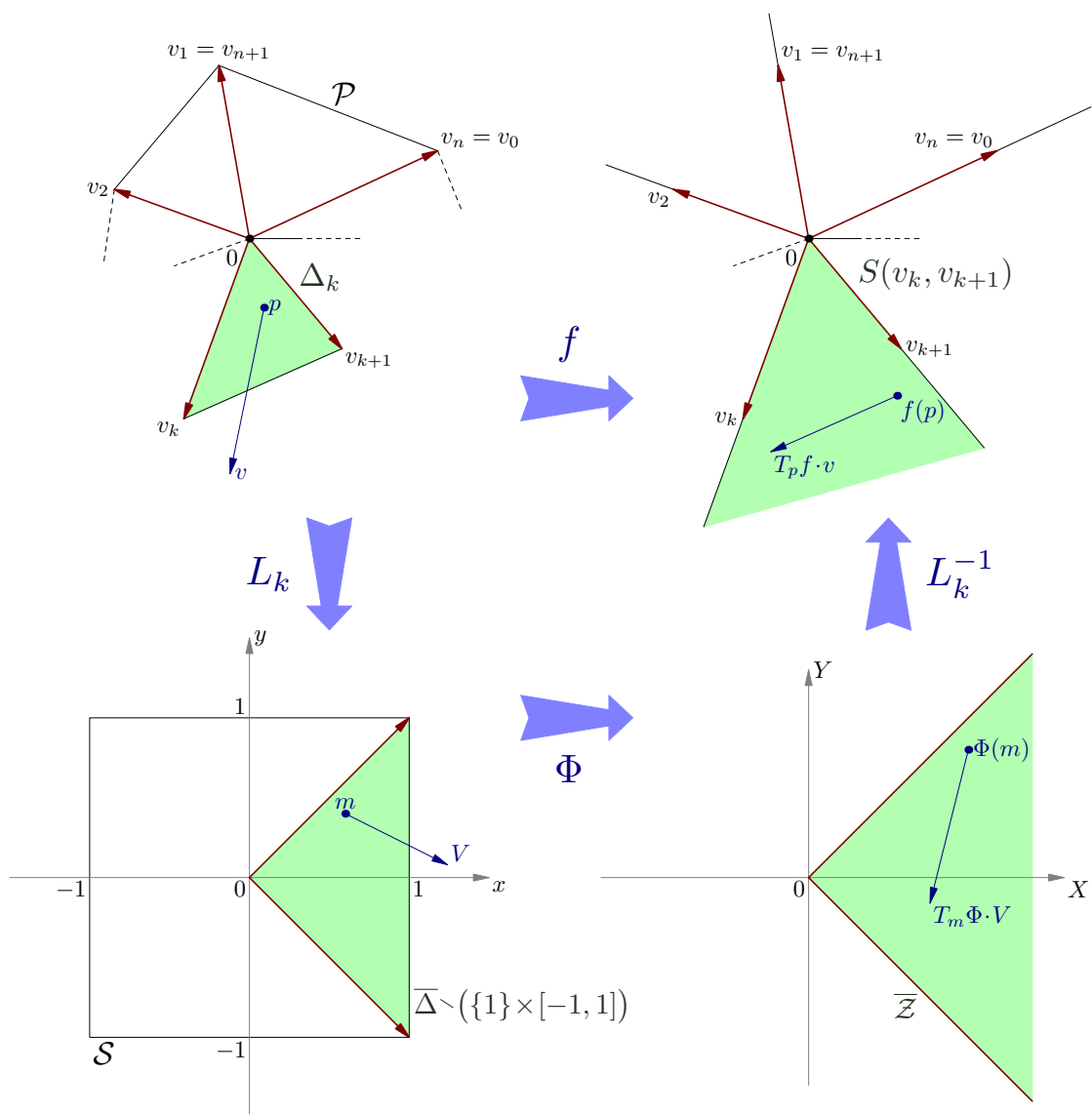


FIGURE 10. The bi-Lipschitz homeomorphism  $f$

for all  $(s, t) \in D$ .

Finally, for all  $0 \leq s < t < 1$ , we have

$$\begin{aligned}
\frac{1}{M} \ln\left(\frac{1-s}{1-t} \times \frac{1+\alpha t}{1+\alpha s}\right) &= \frac{1}{M} \ln\left(\frac{1-s}{1-t}\right) + \frac{1}{M} \ln\left(\frac{1+\alpha t}{1+\alpha s}\right) \\
&\leq \ln\left(\frac{1-s}{1-t}\right) + \frac{1}{M} \ln\left(\frac{1+\alpha t}{1+\alpha s}\right) \quad (\text{since } 1/M \leq 1) \\
&\leq \ln\left(\frac{1-s}{1-t}\right) + \ln\left(\frac{1+t}{1+s}\right) = \ln\left(\frac{1-s}{1-t} \times \frac{1+t}{1+s}\right) \\
&\quad (\text{by the first inequality in Equation 3.1}) \\
&\leq \ln\left(\frac{1-s}{1-t}\right) + M \ln\left(\frac{1+\alpha t}{1+\alpha s}\right) \\
&\quad (\text{by the second inequality in Equation 3.1}) \\
&\leq M \ln\left(\frac{1-s}{1-t}\right) + M \ln\left(\frac{1+\alpha t}{1+\alpha s}\right) \quad (\text{since } M \geq 1) \\
&= M \ln\left(\frac{1-s}{1-t} \times \frac{1+\alpha t}{1+\alpha s}\right)
\end{aligned}$$

This proves Lemma 3.1. □

*Proof of Theorem 3.1.*

• **Point (1):** Let  $g : \mathbf{R}^2 \rightarrow \mathcal{P}$  be the map given by  $g(P) \doteq L_k^{-1}(\Phi^{-1}(L_k(P)))$  for all  $k \in \{1, \dots, n\}$  and  $P \in S(v_k, v_{k+1})$ , this definition making sense since  $\bigcup_{i=1}^n S(v_i, v_{i+1}) = \mathbf{R}^2$ ,  $\Phi^{-1}(L_k(S(v_k, v_{k+1}))) = \overline{\Delta} \setminus (\{1\} \times [-1, 1])$  (using  $L_k(S(v_k, v_{k+1})) = \overline{\mathcal{Z}}$  and point (1) in Proposition 2.1) and  $L_{k-1}^{-1}(\Phi^{-1}(L_{k-1}(P))) = L_k^{-1}(\Phi^{-1}(L_k(P))) \in [0, 1)v_k$  whenever  $P \in \mathbf{R}v_k = S(v_{k-1}, v_k) \cap S(v_k, v_{k+1})$ .

Then it is easy to check that  $f \circ g = \mathfrak{I}_{\mathbf{R}^2}$  and  $g \circ f = \mathfrak{I}_{\mathcal{P}}$  (identity maps), which shows that  $f$  is bijective with  $f^{-1} = g$ .

In addition,  $f$  and  $g$  are continuous since  $L_1, \dots, L_n$  and  $\Phi$  are homeomorphisms.

• **Point (2):** For each  $k \in \{1, \dots, n\}$ , we have  $f([0, 1)v_k) = \mathbf{R}v_k$ , and therefore

$$f(\mathcal{U}) = f(\mathcal{P}) \setminus \left( \bigcup_{k=1}^n f([0, 1)v_k) \right) = \mathbf{R}^2 \setminus \left( \bigcup_{k=1}^n \mathbf{R}v_k \right) = \mathcal{V}.$$

Moreover, since

$$\mathcal{U} = \bigcup_{k=1}^n \overset{\circ}{\Delta}_k \quad \text{and} \quad \mathcal{V} = \bigcup_{k=1}^n \overbrace{S(v_k, v_{k+1})}^{\circ}$$

together with the fact that  $L_1, \dots, L_n$  are smooth by linearity and  $\Phi$  is a smooth diffeomorphism by Proposition 2.1, we get that  $f|_{\mathcal{U}}$  and  $g|_{\mathcal{V}}$  are smooth.

• **Point (3):** Fixing  $k \in \{1, \dots, n\}$  and applying Proposition 2.3 with  $\mathcal{C}_1 \doteq \mathcal{S}$  and  $\mathcal{C}_2 \doteq L_k(\mathcal{P})$ , we get a constant  $B_k \geq 1$  such that for all  $m \in \Delta$  and  $V \in T_m \mathcal{S} = T_m(L_k(\mathcal{P})) = \mathbf{R}^2$ ,

$$\frac{1}{B_k} F_{\mathcal{S}}(m, V) \leq F_{L_k(\mathcal{P})}(m, V) \leq B_k F_{\mathcal{S}}(m, V),$$

and hence

$$\frac{1}{B_k} F_{L_k(\mathcal{P})}(m, V) \leq \|T_m \Phi \cdot V\| \leq 2B_k F_{L_k(\mathcal{P})}(m, V)$$

by Proposition 2.1.

But, since  $L_k$  induces an isometry from  $(\mathcal{P}, d_{\mathcal{P}})$  onto  $(\mathbf{R}^2, d)$  (being affine,  $L_k$  preserves the cross ratio), this is equivalent to saying that for all  $p \in \overset{\circ}{\Delta}_k$  and  $v \in T_p\mathcal{P} = \mathbf{R}^2$  (writing  $m = L_k(p)$  and  $V = L_k(v)$ ), we have

$$\frac{1}{B_k} F_{\mathcal{P}}(p, v) \leq \|T_{L_k(p)}\Phi \cdot L_k(v)\| \leq 2B_k F_{\mathcal{P}}(p, v),$$

which yields

$$\frac{1}{B_k \|L_k\|} F_{\mathcal{P}}(p, v) \leq \|L_k^{-1}(T_{L_k(p)}\Phi \cdot L_k(v))\| = \|T_p f \cdot v\| \leq 2B_k \|L_k^{-1}\| F_{\mathcal{P}}(p, v),$$

where  $\|\cdot\|$  denotes the operator norm on  $\text{End}(\mathbf{R}^2)$  associated with  $\|\cdot\|$ .

Now, if  $K := \max\{B_k \|L_k\| + 2B_k \|L_k^{-1}\| + 1 \mid 1 \leq k \leq n\} \geq 1$ , then

$$(3.2) \quad \frac{1}{K} F_{\mathcal{P}}(p, v) \leq \|T_p f \cdot v\| \leq K F_{\mathcal{P}}(p, v)$$

for all  $p \in \bigcup_{k=1}^n \overset{\circ}{\Delta}_k = \mathcal{U}$  and  $v \in T_p\mathcal{P} = \mathbf{R}^2$ .

We will then prove Theorem 3.1 using the fact that  $(\mathcal{P}, d_{\mathcal{P}})$  and  $(\mathbf{R}^2, d)$  are geodesic metric spaces in which affine segments are geodesics (see Introduction).

Let  $p, q \in \mathcal{P}$  and  $\gamma : [0, 1] \rightarrow \mathcal{P}$  defined by  $\gamma(t) := (1-t)p + tq$ . Assume that  $\gamma|_{[0,1]} \subseteq \mathcal{U}$  and  $q = \gamma(1) \in \bigcup_{k=1}^n [0, 1)v_k$ . The second inequality in Equation 3.2 then implies that for all  $t \in [0, 1)$ , we have

$$d(f(p), f(\gamma(t))) \leq \int_0^t \|T_{\gamma(s)} f \cdot \gamma'(s)\| ds \leq K \int_0^t F_{\mathcal{P}}(\gamma(s), \gamma'(s)) ds = K d_{\mathcal{P}}(p, \gamma(t)),$$

and thus  $d(f(p), f(\gamma(t))) \leq K d_{\mathcal{P}}(p, \gamma(t))$ , which gives

$$(3.3) \quad d(f(p), f(q)) \leq K d_{\mathcal{P}}(p, q)$$

as  $t \rightarrow 1$ .

If now  $p := sv_k$  and  $q := tv_k$  for some  $0 \leq s < t < 1$  and  $k \in \{1, \dots, n\}$ , a straightforward calculation gives

$$\begin{aligned} f(p) = L_k^{-1}(\Phi(sL_k(v_k))) &= L_k^{-1}(\Phi(s(1, -1))) \\ &= L_k^{-1}(\Phi(s, -s)) = L_k^{-1}(\text{atanh}(s), -\text{atanh}(s)) \\ &= \text{atanh}(s)L_k^{-1}(1, -1) = \text{atanh}(s)v_k, \end{aligned}$$

and hence

$$(3.4) \quad d(f(p), f(q)) = \|f(q) - f(p)\| = (\text{atanh}(t) - \text{atanh}(s)) \|v_k\| = \frac{\|v_k\|}{2} \ln\left(\frac{1-s}{1-t} \times \frac{1+t}{1+s}\right),$$

together with

$$e^{2d_{\mathcal{P}}(p,q)} = [v_k, q, p, -t_{\mathcal{P}}^-(0, v_k)v_k] = \frac{1-s}{1-t} \times \frac{t + t_{\mathcal{P}}^-(0, v_k)}{s + t_{\mathcal{P}}^-(0, v_k)},$$

or equivalently

$$(3.5) \quad d_{\mathcal{P}}(p, q) = \frac{1}{2} \ln\left(\frac{1-s}{1-t} \times \frac{1 + t\alpha_k}{1 + s\alpha_k}\right),$$

where  $\alpha_k := 1/t_{\mathcal{P}}^-(0, v_k) > 0$ .

Then, using Lemma 3.1 with  $\alpha = \alpha_k$  and denoting  $\Lambda_k = M(\alpha_k) \times \max\{\|v_k\|, 1/\|v_k\|\} \geq 1$ , Equations 3.4 and 3.5 yield

$$(3.6) \quad \frac{1}{\Lambda_k} d_{\mathcal{P}}(p, q) \leq d(f(p), f(q)) \leq \Lambda_k d_{\mathcal{P}}(p, q).$$

If now  $p$  and  $q$  are arbitrary chosen in  $\mathcal{P}$ , the closed affine segment joining  $p$  and  $q$  either meets  $\bigcup_{k=1}^n [0, 1)v_k$  in *at most*  $n$  points, or has at least two distinct points in common with some  $[0, 1)v_k$  for  $k \in \{1, \dots, n\}$ .

Therefore, it follows from Equation 3.3 and the second inequality in Equation 3.6 that  $d(f(p), f(q)) \leq C d_{\mathcal{P}}(p, q)$  holds with  $C = \max\{\Lambda_k \mid 1 \leq k \leq n\} + K \geq 1$ .

Finally, using the first inequalities in Equations 3.2 and 3.6, the same arguments for  $f^{-1}$  as those for  $f$  lead to  $d_{\mathcal{P}}(p, q) \leq C d(f(p), f(q))$  for any  $p, q \in \mathcal{P}$ .

This ends the proof of Theorem 3.1. □

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