LIPSCHITZ CHARACTERISATION OF POLYTOPAL HILBERT GEOMETRIES

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Abstract. We prove that the Hilbert Geometry of a convex set is bi-lipschitz equivalent to a normed vector space if and only if the convex is a polytope.

Introduction and statement of results

A Hilbert geometry is a particularly simple metric space on the interior of a compact convex set $\mathcal{C}$ modelled on the construction of the Klein model of Hyperbolic geometry inside an euclidean ball. This metric happens to be a complete Finsler metric whose set of geodesics contains the straight lines. Since the definition of the Hilbert geometry only uses cross-ratios, the Hilbert metric is a projective invariant.

In addition to ellipsoids, a second family of convex sets play a distinct role among Hilbert geometries: the simplexes. If the ellipsoids' geometry is isometric to the Hyperbolic geometry and are the only Riemannian Hilbert geometries (see D.C. Kay [Kay67, Corollary 1]), at the opposite side simplexes happen to be the only ones whose geometry is isometric to a normed vector space (e.g. see De la Harpe [dlH93] for the existence and Foertsch & Karlsson [FK05] for the uniqueness).

A lot of the recent works done in the context of these geometries focus on finding out how close they are to the hyperbolic geometry, from different viewpoints (see, e.g., A. Karlsson & G. Noskov [KN02], Y. Benoist [Ben03, Ben06] for δ-hyperbolicity, E. Socie-Methou [SM02, SM04] for automorphisms and B. Colbois & C. Vernicos [CV06, CV07] for the spectrum). It is now quite well understood that this is closely related to regularity properties of the boundary of the convex set. For instance if the boundary is C^2 with positive Gaussian curvature, then B. Colbois & P. Verovic [CV04] have shown that the Hilbert geometry is bi-lipschitz equivalent to the Hyperbolic geometry.

The present work investigates those Hilbert geometries close to a norm vector space.

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Along that path it has been noticed than any polytopal Hilbert geometry can be isometrically embedded in a normed vector space of dimension twice the number of it faces (see B.C. Lins [Lin07]). Then B. Colbois & P. Verovic [CV] showed that in fact no other Hilbert geometry could be quasi-isometrically embedded into a normed vector space. Furthermore with B. Colbois and P. Verovic [CVVa] we have shown that the Hilbert geometries of plane polygons are bi-lipschitz to the euclidean plane. Even though we saw no reason for this result not to hold in higher dimension, our point of view made it difficult to obtain a generalisation due to the computations it involved. The present works aims at filling that gap by giving a slightly different proofs which holds in all dimension, with less computations, but at the cost of a longer study of simplexes. Hence our main results is the following,

Theorem 1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex polytope, its Hilbert Geometry $(\mathcal{P}, d_{\mathcal{P}})$ is bi-lipschitz to the n-dimensional euclidean geometry $(\mathbb{R}^n, \|\cdot\|)$. In other words there exist a map $F\colon \mathcal{P} \to \mathbb{R}^n$ and a constant L such that for any two points x and y in $\mathcal{P}$,*

$$\frac{1}{L} \cdot \big\|F(x) - F(y)\big\| \leqslant d_{\mathcal{P}}(x, y) \leqslant L \cdot \big\|F(x) - F(y)\big\|.$$

The main idea is that a polytopal convex set can be decomposed into pyramids with apex its barycentre and base its faces, and then to prove that each pyramid is bi-Lipschitz to the cone it defines. However due to the multitude of available faces in dimension higher than two, a reduction is needed and consists in using the barycentric subdivision to decompose each of these pyramids into similar simplexes, and to prove that each of these simplexes is bi-Lipschitz to the cone it defines.

The following corollary "à la" Bourbaki sums up the known characterisations of the polytopal Hilbert geometries

Corollary 2. *Let $\mathcal{C} \in \mathbb{R}^n$ be a properly open convex set and $(\mathcal{C}, d_{\mathcal{C}})$ its Hilbert geometry. Then the following are equivalent*

(1) $\mathcal{C}$ is a polytopal convex domain;
(2) $(\mathcal{C}, d_{\mathcal{C}})$ is bi-lipschitz equivalent to an n-dimensional vector space;
(3) $(\mathcal{C}, d_{\mathcal{C}})$ is quasi-isometric to the euclidean n-dimensional vector space;
(4) $(\mathcal{C}, d_{\mathcal{C}})$ isometrically embeds into a normed vector space;
(5) $(\mathcal{C}, d_{\mathcal{C}})$ quasi-isometrically embeds into a normed vector space;

Note. Theorem 1 was found and proved with a completely different approach by Andreas Bernig [AB09]. The two approaches are somewhat dual to one another: where Bernig uses faces, we use vertices. Let us also stress out that the heart of our proof lies on the comparison Theorem 8 which is interesting on its own.

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1. Definition of a Hilbert Geometry and notations

1.1. **Hilbert Geometries.** Let us recall that a Hilbert geometry $(\mathcal{C}, d_{\mathcal{C}})$ is a non empty bounded open convex set $\mathcal{C}$ on $\mathbb{R}^n$ (that we shall call *convex domain*) with the Hilbert distance $d_{\mathcal{C}}$ defined as follows : for any distinct points p and q in $\mathcal{C}$, the line passing through p and q meets the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ at two points a and b, such that one walking on the line goes consecutively by a, p, q b (figure 1). Then we define

$$d_{\mathcal{C}}(p,q) = \frac{1}{2}\ln[a,p,q,b],$$

where $[a,p,q,b]$ is the cross ratio of (a,p,q,b), i.e.,

$$[a,p,q,b] = \frac{\|q-a\|}{\|p-a\|} \times \frac{\|p-b\|}{\|q-b\|} > 1,$$

with $\|\cdot\|$ the canonical euclidean norm in $\mathbb{R}^n$.

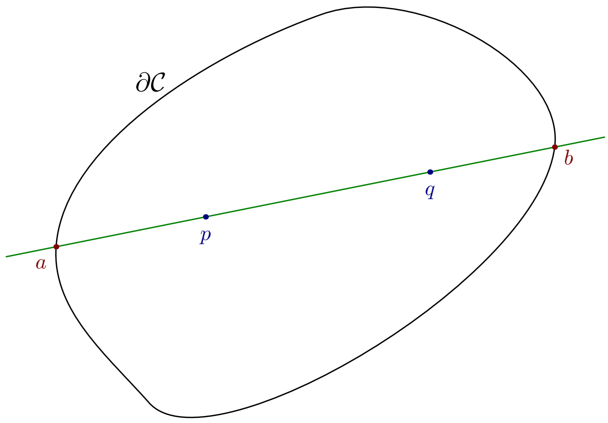

Figure 1. The Hilbert distance

Note that the invariance of the cross-ratio by a projective map implies the invariance of $d_{\mathcal{C}}$ by such a map.

These geometries are naturally endowed with a C^0 Finsler metric $F_{\mathcal{C}}$ as follows: if $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^n$ with $v \neq 0$, the straight line passing by p and directed by v meets $\partial\mathcal{C}$ at two points p^+ and p^- ; we then define

$$F_{\mathcal{C}}(p,v) = \frac{1}{2}\|v\|\left(\frac{1}{\|p-p^-\|} + \frac{1}{\|p-p^+\|}\right) \quad \text{and} \quad F_{\mathcal{C}}(p,0) = 0.$$

The Hilbert distance $d_{\mathcal{C}}$ is the length distance associated to $F_{\mathcal{C}}$.

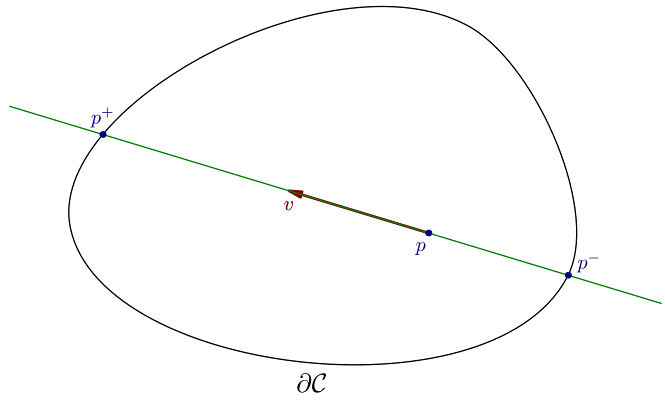

FIGURE 2. The Finsler structure

1.2. **Faces.** Recall that to a closed convex set K we can associate an equivalent relation, stating that two points A and B are equivalent if there exists a segment $[C, D] \subset K$ containing the segment $[A, B]$ such that $C \neq A, B$ and $D \neq A, B$. The equivalent classes are called *faces*. A face is called a *k-face*, if the dimension of the smallest affine space containing it is k. As usual we call *vertex* a 0-dimensional face.

In this paper a simplex in $\mathbb{R}^n$ is the convex closure of $n+1$ projectively independent points, i.e., a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3$, etc...

Definition 3 (Conical faces)**.** Let $\mathcal{C}$ be a convex set. If there is a simplex S containing $\mathcal{C}$ such that a non-empty k-face $f \subset \partial\mathcal{C}$ is included in a k-face of S, we say that f is a *conical face* and that $\mathcal{C}$ admits a conical face.

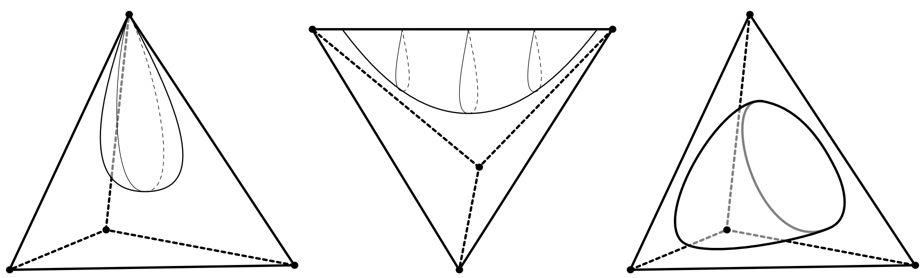

FIGURE 3. Conical faces in dimension 3

When a face f is in the boundary of another face F we write $f < F$.

Definition 4 (Conical flag)**.** Let $\mathcal{C}$ be a convex set in $\mathbb{R}^n$. If there exist a simplex $\mathcal{S}$ contained in $\mathcal{C}$, with a flag $\emptyset < f_0 < f_1 < f_2 < \ldots < f_{n-1} < \mathcal{S}$, such that for any $k = 0, \ldots, n-1$,

(1) f_k is inside a k-conical face of $\mathcal{C}$;
(2) no other k-face of $\mathcal{S}$ is inside a k-conical face of $\mathcal{C}$;

then we will call $f_0 < f_1 < f_2 < \ldots < f_{n-1} < \mathcal{C}$ a *conical flag* and say that $\mathcal{C}$ admits a *conical flag*. Furthermore we will call $\mathcal{S}$ a *conical flag neighborhood*.

1.3. **Prismatic neighborhoods and cones.**

Definition 5 (Prismatic neighborhoods)**.** Let $\mathcal{S}$ be a simplex in $\mathbb{R}^n$ and let x_k be a point inside a k-face of $\mathcal{S}$. Let A_k be the k-dimensional affine space containing x_k and its k-face. Let $(e_1, e_2, \ldots, e_k)$ be an orthonormal basis of the vector space $A_k - x_k$ completed into an orthonormal basis of $\mathbb{R}^n$ with $v_1, \ldots, v_{n-k}$, where each of these vectors is parallel to one of the $k+1$-faces of $\mathcal{S}$ which contain x_k in their boundary.

- An (ε, α)-prismatic neighborhood with k-dimensional apex of x_k is the convex closure of a k-cube of diameter $2\sqrt{k}\varepsilon$ centred at x_k in A_k and its translate by αv_i, $i = 1, \ldots, n-k$.

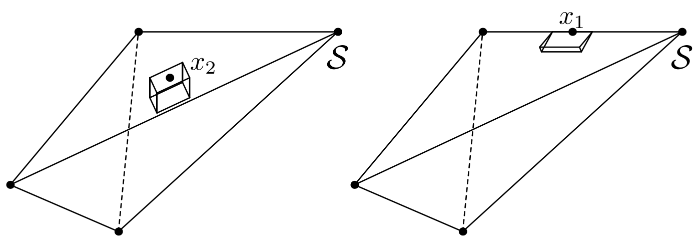

FIGURE 4. Prismatic neighborhoods of a point in a 1-face and a 2-face in dimension 3

- An ε-prismatic cone with k-dimensional apex centred at x_k is the union of all (ε, α)-prismatic neighborhoods with k-dimensional apex of x_k for $\alpha \in \mathbb{R}^+$.

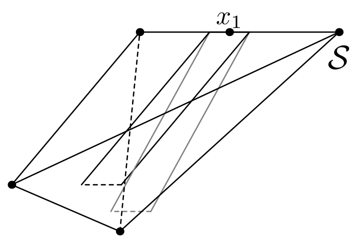

FIGURE 5. Prismatic cone of a 1-face in dimension 3

The following lemma, which compares the Hilbert geometries of a prismatic neighborhood of a point x and its corresponding cone around that point x, will play a critical role in the sequel.

Lemma 6. *Let $\mathcal{S}$ be a simplex in $\mathbb{R}^n$ and let x_k be a point inside a k-face of $\mathcal{S}$. For any pair of positive numbers $\epsilon, \alpha > 0$ let $\mathcal{P}_k$ be a*

(ε,α)-prismatic neighborhood with k-dimensional apex of x_k, and $\mathcal{PC}_k$ the corresponding prismatic cone. Then for any sequence $(y_n,v_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_k\times\mathbb{R}^n$ such that the sequence $(y_n)_{n\in\mathbb{N}}$ tends to x_k, one has

$$\lim_{n\to\infty}\frac{F_{\mathcal{P}_k}(y_n,v_n)}{F_{\mathcal{PC}_k}(y_n,v_n)}=1.$$

This lemma is a straightforward consequence of proposition 2.6's proof in [BBV10] which can be restated in the following way

Proposition 7. *Let K,K' be closed convex sets not containing any straight line and for any point x in $K\cap K'$, let $\|\cdot\|_x$, $\|\cdot\|'_x$ be their respective Finsler norm induced by the their respective Hilbert geometries. Let $p\in\partial K$, E_0 a support hyperplane of K at p and E_1 a hyperplane parallel to E_0 intersecting K. Suppose that K and K' have the same intersection with the strip between E_0 and E_1 (in particular $p\in\partial K'$). Then as functions on $\mathbb{R}P^{n-1}$, $\|\cdot\|_x/\|\cdot\|'_x$ uniformly converge to 1.*

2. Metric comparison around a conical flag

Theorem 8. *Let $\mathcal{A}$ and $\mathcal{B}$ bet two convex set with a common conical flag neighborhood $\mathcal{S}$ then there exists a constant C such that for any $x\in\mathcal{S}$ and $v\in\mathbb{R}^n$ one has*

$$\frac{1}{C}\cdot F_{\mathcal{B}}(x,v)\leqslant F_{\mathcal{A}}(x,v)\leqslant C\cdot F_{\mathcal{B}}(x,v) \tag{1}$$

Example 9. In the two dimensional case the condition is that $\mathcal{A}$ and $\mathcal{B}$ contain a triangle $\mathcal{S}$, one of its edges on their boundaries, a vertex of which, and only one, is an extremal point of both of them where they fail to be C^1.

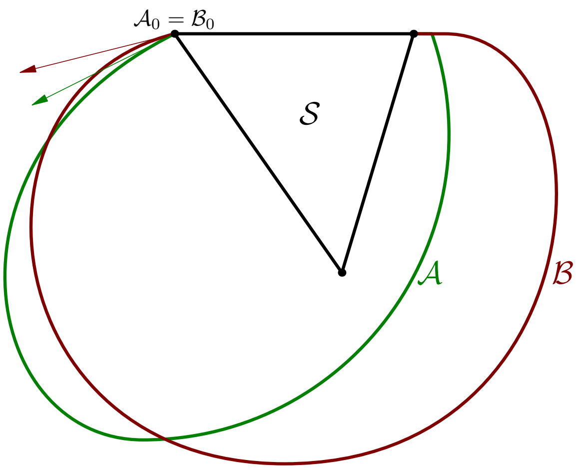

Figure 6. Illustration of Example 9

To prove Theorem 8 we will reduce to the case where both $\mathcal{A}$ and $\mathcal{B}$ are simplexes and $\mathcal{A} \subset \mathcal{B}$ (see Figure 7). This is the intermediate Lemma 10 whose statement and illustration follow.

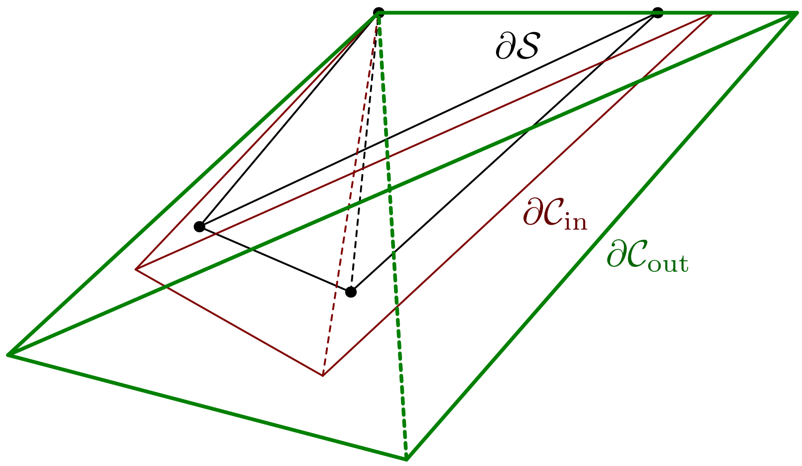

FIGURE 7. The simplexes of lemma 10

Lemma 10. *Suppose that $\mathcal{S}$, $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$ are three n-simplexes such that $\mathcal{S} \subset \mathcal{C}_{in} \subset \mathcal{C}_{out}$ and $\mathcal{S}$ is a common conical flag neighborhood of $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$. Then there exists a constant M such that for any $x \in \mathcal{S}$ and any vector $v \in \mathbb{R}^n$ one has*

$$F_{\mathcal{C}_{out}}(x,v) \leqslant F_{\mathcal{C}_{in}}(x,v) \leqslant M \cdot F_{\mathcal{C}_{out}}(x,v)$$

We can now present Theorem 8's proof as a corollary.

Proof of Theorem 8. Thanks to our assumption we can built a simplex $\mathcal{C}_{\text{in}}$ inside $\mathcal{A} \cap \mathcal{B}$ containing $\mathcal{S}$ and a simplex $\mathcal{C}_{\text{out}}$ containing $\mathcal{A} \cup \mathcal{B}$ satisfying the assumptions required by lemma 10.

Then the inclusions $\mathcal{C}_{\text{in}} \subset \mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B} \subset \mathcal{C}_{\text{out}}$ give the following sets of inequalities

$$F_{\mathcal{C}_{\text{out}}}(x,v) \leqslant F_{\mathcal{A}}(x,v) \leqslant F_{\mathcal{C}_{\text{in}}}(x,v),$$

and

$$F_{\mathcal{C}_{\text{out}}}(x,v) \leqslant F_{\mathcal{B}}(x,v) \leqslant F_{\mathcal{C}_{\text{in}}}(x,v).$$

We combine these inequalities to obtain

$$\frac{F_{\mathcal{C}_{\text{out}}}(x,v)}{F_{\mathcal{C}_{\text{in}}}(x,v)} \leqslant \frac{F_{\mathcal{A}}(x,v)}{F_{\mathcal{B}}(x,v)} \leqslant \frac{F_{\mathcal{C}_{\text{in}}}(x,v)}{F_{\mathcal{C}_{\text{out}}}(x,v)},$$

and we conclude thanks to Lemma 10.

Let us make the construction of $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ precise. To do so, let us consider the conical flag $f_0 < f_1 < \cdots < f_{n-1} < S$. Then we will denote the k-face of $\mathcal{A}$ containing f_k by $\mathcal{A}_k$ and similarly by $\mathcal{B}_k$ the corresponding face of $\mathcal{B}$.

For $n \geqslant k \geqslant 0$, let us denote by v_k the vertex of $\mathcal{S}$ inside $\mathcal{A}_k \cap \mathcal{B}_k$, but not inside $\mathcal{A}_{k-1} \cap \mathcal{B}_{k-1}$ and by p_k the barycentre of the vertexes $v_k, \dots, v_0$. Then by assumption there exists a point $v_k \neq v_{k,1} \in \mathcal{A}_k \cap \mathcal{B}_k$ such that the segment $[p_k, v_{k,1}]$ contains v_k. We take for $\mathcal{C}_{\text{in}}$ the convex hull of $v_{n,1}, \dots, v_{0,1}$.

For $\mathcal{C}_{\text{out}}$, we will actually built its convex dual. Indeed, if we take the dual convexes of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ with respect to some point in the interior of $\mathcal{S}$, we obtain respectively three convex sets $\mathcal{B}^*$, $\mathcal{A}^*$ and $\mathcal{S}^*$ such that both $\mathcal{B}^*$ and $\mathcal{A}^*$ are inside $\mathcal{S}^*$. Furthermore, for $k = 0, \cdots, n-1$, let us denote by $\mathcal{S}_k^*$ the k-face of $\mathcal{S}^*$ corresponding to the hyperplanes tangent to f_{n-k-1}. Then by assumption, $\mathcal{S}_k^*$ contains the hyperplanes tangent to $\mathcal{A}_{n-k-1}$ and to $\mathcal{B}_{n-k-1}$ but not to $\mathcal{A}_{n-k}$ or $\mathcal{B}_{n-k}$.

Let us also remark that as f_{n-k-1} is in the intersection of $\mathcal{A}_{n-k-1}$ and $\mathcal{B}_{n-k-1}$, which are both conical faces of $\mathcal{A}$ and $\mathcal{B}$ respectively. Therefore the intersection of hyperplanes containing both these faces but no other faces of either $\mathcal{A}$ or $\mathcal{B}$, and simultaneously tangent to $\mathcal{A}$ and $\mathcal{B}$ is an open and not empty set of $\mathcal{S}_k^*$, which we shall denote by $\mathcal{O}_k^*$.

In particular the vertex $\mathcal{S}_0^* = \mathcal{O}_k^*$ corresponds to the common hyperplane containing the three faces $\mathcal{A}_{n-1}$, $\mathcal{B}_{n-1}$ and f_{n-1}.

Now, let w_0 be the vertex $\mathcal{S}_0^*$, and for $k = 1, \dots, n-1$ take a point w_k in $\mathcal{O}_k$. Let also take a point w_n in the intersection of the convex sets $\mathcal{A}^*$ and $\mathcal{B}^*$. Then by construction, if we let $\mathcal{C}_{\text{out}}^*$ be the interior of the convex hull of $w_0, \cdots, w_n$, it is a simplex, which is a common conical flag of $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{S}^*$. Thus its dual will contain both $\mathcal{A}$ and $\mathcal{B}$, and admits $\mathcal{S}$ as a conical flag neighborhood. Therefore that is our simplex $\mathcal{C}_{\text{out}}$.

$\square$

2.1. **Proof of lemma 10.**

2.1.1. *Notation needed along the proof.* Recall that $\mathcal{S}$, $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ are three n dimensional simplexes such that $\mathcal{S} \subset \mathcal{C}_{\text{in}} \subset \mathcal{C}_{\text{out}}$. By assumption the closure of one of the $n-1$ dimensional faces of $\mathcal{S}$ is the intersection of the closure of these three simplexes. In fact, for every $k \leqslant n-1$, there is a unique k dimensional face of $\mathcal{S}$, denoted by f_k, which is also inside a k dimensional face of $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$, denoted respectively by $\varphi_{\text{in},k}$ and $\varphi_{\text{out},k}$. Our assumptions imply

$$f_k \subset \varphi_{\text{in},k} \subset \varphi_{\text{out},k}.$$

We will denote by A_k the k-dimensional affine space containing the three faces f_k, $\varphi_{\text{in},k}$ and $\varphi_{\text{out},k}$ for $0 \leqslant k \leqslant n$. Hence A_0 is a common vertex to the three simplexes and A_n the whole space $\mathbb{R}^n$.

2.1.2. *Step 0: Ignition.* The left inequality of lemma 10 is a straightforward consequence of the fact that $\mathcal{C}_{\text{in}} \subset \mathcal{C}_{\text{out}}$.

For the right inequality, by homogeneity we can restrict to vectors v in the unit euclidean sphere $\mathcal{B}_n$. Hence we will focus on the following ratio, where x is inside $\mathcal{S}$ and v a unit vector

$$\mathcal{Q}(x,v) = \frac{F_{\mathcal{C}_{\text{in}}}(x,v)}{F_{\mathcal{C}_{\text{out}}}(x,v)}.$$

We want to show that $\mathcal{Q}$ remains bounded on $\mathcal{S} \times \mathcal{B}_n$

Hypothesis. Let us suppose by contradiction that $\mathcal{Q}$ is not bounded.

Thanks to that hypothesis we can find a sequence $(x_l, v_l)_{l\in\mathbb{N}}$ such that for all $l \in \mathbb{N}$, $x_l \in \mathcal{S}$, $v_l \in \mathcal{B}_n$ and most importantly

$$\mathcal{Q}(x_l, v_l) \to +\infty. \tag{2}$$

Due to the compactness of $\overline{\mathcal{S}} \times \mathcal{B}_n$, at the cost of taking a sub-sequence, we can assume that this sequence converges to (x_∞, v_∞)

Remark 11. *If x remains in a compact set U inside $\mathcal{C}_{in}$, then $\mathcal{Q}$ remains bounded as a continuous function of two variables over the compact set $U \times \mathcal{B}_n$.*

2.1.3. *Step 1: Focusing on faces.* Thanks to the above remark 11, if $(x_l)_{l\in\mathbb{N}}$ were to converge toward a point in $\mathcal{C}_{\text{in}}$, we would get a contradiction. Hence x_∞ has to be on the boundary of $\mathcal{C}_{\text{in}}$, which implies that x_∞ is on a common face of the three simplexes.

We will suppose that x_∞ belongs to the k-dimensional face f_k of $\mathcal{S}$ and obtain a contradiction.

To do so we will make two simplifications:

1. We first replace the three simplexes by three prismatic neighborhoods of x_∞, is such a way that $\mathcal{Q}(x_l, v_l)$ remains bounded if the analogous quotient for these prismatic neighborhoods does (Step 2 and 3).
2. We then replace the three prismatic neighborhoods by their three corresponding prismatic cones centred at x_∞ and then we prove that the corresponding quotient remains bounded (Step 4, lemmata 6 and 13).

2.1.4. *Step 2: The prismatic neighborhoods.* For the following constructions we fix k and we suppose that the limit point x_∞ belongs to the k-dimensional face of $\mathcal{S}$, i.e., $x_\infty \in f_k$.

If $k \neq 0$, choose $0 < \alpha < \beta < \gamma$ such that

(i) the (α, α)-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$ is inside $\mathcal{S}$;
(ii) the (β, β)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\text{in}}$ is inside $\mathcal{C}_{\text{in}}$;
(iii) the (γ, γ)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\text{out}}$ **contains** $\mathcal{C}_{\text{out}}$;
(iv) the (β, β)-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\text{in}}$ contains the (α, α)-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$.

Then we will denote by

(i) $\mathcal{P}_{\mathcal{S},k}$ the $(\alpha/2, \alpha/2)$-prismatic neighborhood of x_∞ with respect to $\mathcal{S}$;
(ii) $\mathcal{P}_{\text{in},k}$ the $(\beta/2, \beta/2)$-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\text{in}}$;
(iii) $\mathcal{P}_{\text{out},k}$ the $(2\gamma, 2\gamma)$-prismatic neighborhood of x_∞ with respect to $\mathcal{C}_{\text{out}}$;

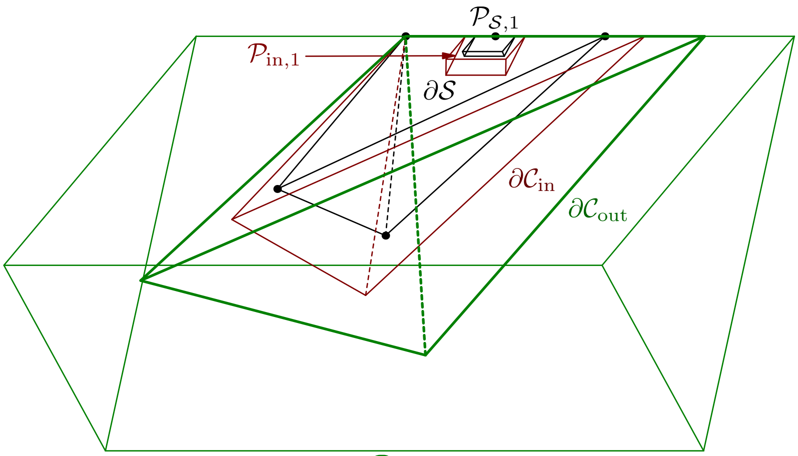

Figure 8. Prismatic neighborhoods of a 1-face in dimension 3

For $k = 0$, we take $\mathcal{S} = \mathcal{P}_{\mathcal{S},0}$, $\mathcal{C}_{\text{in}} = \mathcal{P}_{\text{in},0}$ and $\mathcal{C}_{\text{out}} = \mathcal{P}_{\text{out}.0}$.

Now let us define for any point x in the prismatic neighborhood of x_∞ with respect to $\mathcal{S}$, $\mathcal{P}_{\mathcal{S},k}$, and any unit vector $v \in \mathcal{B}_n$ the following ratio

$$R_k(x, v) = \frac{F_{\mathcal{P}_{\text{in},k}}(x, v)}{F_{\mathcal{P}_{\text{out},k}}(x, v)}. \tag{3}$$

We introduce this ratio because for any point $x \in \mathcal{P}_{\mathcal{S},k}$ and vector v it bounds from above $\mathcal{Q}$, i.e.:

$$\mathcal{Q}(x, v) \leqslant R_k(x, v). \tag{4}$$

2.1.5. *Step 3: The prismatic cones.* Let us denote by

(i) $\mathcal{PC}_{\mathcal{S},k}$ the $\alpha/2$-prismatic cone centred at x_∞ with respect to $\mathcal{S}$;
(ii) $\mathcal{PC}_{\text{in},k}$ the $\beta/2$-prismatic cone centred at x_∞ with respect to $\mathcal{C}_{\text{in}}$;
(iii) $\mathcal{PC}_{\text{out},k}$ the 2γ-prismatic cone centred at x_∞ with respect to $\mathcal{C}_{\text{out}}$.

by construction we have $\mathcal{PC}_{\mathcal{S},k} \subset \mathcal{PC}_{\text{in},k} \subset \mathcal{PC}_{\text{out},k}$.

Finally we associate the following ratio with these prismatic cones.

$$\mathcal{R}_k(x, v) = \frac{F_{\mathcal{PC}_{\text{in},k}}(x, v)}{F_{\mathcal{PC}_{\text{out},k}}(x, v)}. \tag{5}$$

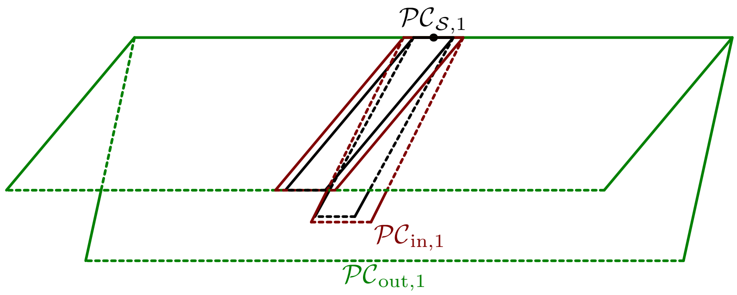

FIGURE 9. Prismatic cones of a 1-face in dimension 3

2.1.6. *Step 4: Comparisons.* First notice that there exist an integer N such that for all $l > N$, x_l will be inside $\mathcal{P}_{\mathcal{S},k}$. Hence, applying lemma 6 we get the following equivalence.

Lemma 12. *Let us fix $0 \leqslant k \leqslant n$ and let (y_l, w_l) be a sequence with y_l in the interior prismatic cone $\mathcal{PC}_{\mathcal{S},k}$ converging toward (x_∞, w_∞) with $w_\infty \in \mathcal{B}_n$, then*

$$\lim_{l\to\infty} \frac{R_k(x_l, v_l)}{\mathcal{R}_k(x_l, v_l)} = 1$$

The previous lemma 12 allows us to focus on the prismatic cones, therefore the heart of our proof now lies in the following key lemma.

Lemma 13. *Let us fix $0 \leqslant k \leqslant n$ and let (y_l, w_l) be a sequence with y_l in the interior prismatic cone $\mathcal{PC}_{\mathcal{S},k}$ converging toward (x_∞, w_∞) with $w_\infty \in \mathcal{B}_n$, then there is a constant c such that for all $l \in \mathbb{N}$ one has*

$$\mathcal{R}_k(y_l, w_l) \leqslant c. \tag{6}$$

Proof of Lemma 13. We suppose that x_∞ is the origin thus the affine k-dimensional subspace A_k containing f_k is actually a sub-vector space. We then consider the decomposition

$$\mathbb{R}^n = A_k \oplus A_k^\perp,$$

and the *vectorial affinity* VA_λ which is defined as the identity on A_k and as the dilation of ratio λ on $A_k^\perp$. When $k = 0$ this is just a dilation centred at the origin.

The three prismatic cones are invariant by these vectorial affinities, hence VA_λ is an isometry with respect to their Hilbert Geometries.

Now consider a supporting hyperplane E_0 to these prismatic cones at the origin, and an affine hyperplane E_1 parallel to E_0 intersecting the prismatic cones and the face f_{k+1}. Then for any l, there is a λ such that y_l is is pushed away from the origin onto the hyperplanes E_1 while staying in the interior of the inside prismatic cone $\mathcal{PC}_{\mathcal{S},k}$, i.e.

$$\exists \lambda, VA_\lambda(y_l) \in E_1 \text{ and } VA_\lambda(y_l) \in \mathcal{PC}_{\mathcal{S},k}.$$

This gives a new sequence (y'_l, w'_l), with $w'_l = VA_\lambda(w_l)/||VA_\lambda(w_l)||$, which stays in the hyperplane E_1, and such that $\mathcal{R}_k(y_l, w_l) = \mathcal{R}_k(y'_l, w'_l)$.

Case $k = n$: In that situation, the new sequence remains in the intersection of E_1 with the interior prismatic cone $\mathcal{PC}_{\mathcal{S},n}$, which is a common compact set of the prismatic cones $\mathcal{P}_{\text{in},n}$ and $\mathcal{P}_{\text{out},n}$, and thus we conclude thanks to remark 11, that the ratio $\mathcal{R}_n(y_l, w_l)$ remains bounded.

Case $k < n$: By descending induction, suppose that for any triple of prismatic cones with k'-dimensional apex of type $\mathcal{PC}_{*,k'}$ which can occur in a construction in step 3, with $k' > k$, our conclusion holds. The initial step was done in the previous case.

Regarding the sequence (y'_l, w'_l): either it stays away from the common hyperplane A_{n-1}, which means that the sequence remains in a common compact set of the prismatic cones $\mathcal{P}_{\text{in},k}$ and $\mathcal{P}_{\text{out},k}$, and thus by remark 11 we conclude once again that there is a constant c such that

$$\mathcal{R}_k(y_l, w_l) = \mathcal{R}_k(y'_l, w'_l) \leqslant c,$$

or the sequence admits a sub-sequence converging to the common hyperplane A_{n-1} while remaining in the hyperplane E_1, hence away from A_k. Without loss of generality we thus can suppose that the whole sequence (y'_l, w'_l) converges to (y_∞, w_∞), with y_∞ in some common k'-dimensional, with $k' > k$ face of the three prismatic cones.

Remark that from a projective point of view, the prismatic cones are actually prismatic polytopes, having another common face, the projective hyperplane at infinity. In other words, up to a change of affine chart, which is an isometry for the respective Hilbert Geometries, we can suppose that the prismatic cones are prismatic polytopes.

Once we remarked this, we can now build three new prismatic polytopes of type $\mathcal{P}_{*,k'}$ and their corresponding prismatic cones of type $\mathcal{P}_{*,k'}$ containing y_∞, obtaining a new ratio of type $\mathcal{R}_{k'}$ which bounds from above our ratio $\mathcal{R}_k$. Now the induction assumption allows us to conclude that this new ratio is bounded from above, and therefore $\mathcal{R}_k(y_l, w_l)$ also stays bounded from above as l goes to infinity and our proof is completed. □

2.1.7. *Step 5: Conclusion.* Let us consider a converging sub-sequence $(x_l, v_l)_{l\in\mathbb{N}}$ satisfying the divergence (2). Then for some $0 \leqslant k \leqslant n$, the limit x_∞ belongs to the face f_k.

Therefore lemmata 12 and 13 imply that $R_k(x_l, v_l)$ remains bounded as $l \to \infty$, and by the inequality 4 that $\mathcal{Q}(x_l, v_l)$ as well, which is absurd.

Hence our initial hypothesis, that $\mathcal{Q}$ is not bounded is false, which concludes our proof.

3. Polytopal Hilbert geometries are bi-lipschitz to euclidean vector spaces

> The barycentre of a polytope and its faces induce a decomposition of the polytope into pyramids with apex the barycentre and base the faces. These pyramids also give rise to cones with summit their apex which in turn decompose the ambient space. We built a map which sends these pyramids to their corresponding cones and which is a bi-lipschitz map between the Hilbert geometry of the polytope and the Euclidean geometry of the ambient space.

The proof consists in building a bi-lipschitz map and goes along the following steps:

(1) Using the barycentric subdivision, in section 3.1 we decompose a polytopal domain of $\mathbb{R}^n$ into a finite number of simplexes S_i, which we call *barycentric simplexes*.
(2) In section 3.2 we prove that each barycentric simplex S_i of a polytope admits a bi-lipschitz embedding onto a barycentric simplex $\mathcal{S}_n$ of the n-simplex.
(3) We show that we can send isometrically the barycentric simplex of a n-simplex onto a cone of a vector space W_n, using a known isometric map between the n-simplex and W_n (see section 3.3). This cone is then sent isometrically to the cone associated to a barycentric simplex of a polytope.
(4) Finally this allows us in section 3.4 to define a map from the polytopal domain to $\mathbb{R}^n$ by patching the bi-lipschitz embeddings associated to each of its barycentric simplexes.

3.1. Cell decomposition of the polytope.

Consider $\mathcal{P}$ a polytope in $\mathbb{R}^n$. We will denote by f_{ij} the i^{th} face of dimension $1 \leqslant j \leqslant n$.

Let p_n be the barycentre of $\mathcal{P}$, and p_{ij} be the barycentre of the face f_{ij}. Let us denote by D_{ij} the half line from p_n to p_{ij}.

We recall the following well known property, emphasizing an aspect we need.

Property 14. *A polytopal domain $\mathcal{P}$ in $\mathbb{R}^n$ can be uniquely decomposed as a union of n-dimensional simplexes, called* barycentric simplexes *or* cells, *such that the vertexes are barycentre of the faces and each cell is a conical flag neighborhood of the polytope $\mathcal{P}$.*

In the sequel let us adopt the following notations and conventions: If $\mathcal{P}$ is a polytope in $\mathbb{R}^n$, we will suppose that its barycentre is the origin and denote by S_i for $i = 1, \dots, N$, its barycentric simplexes.

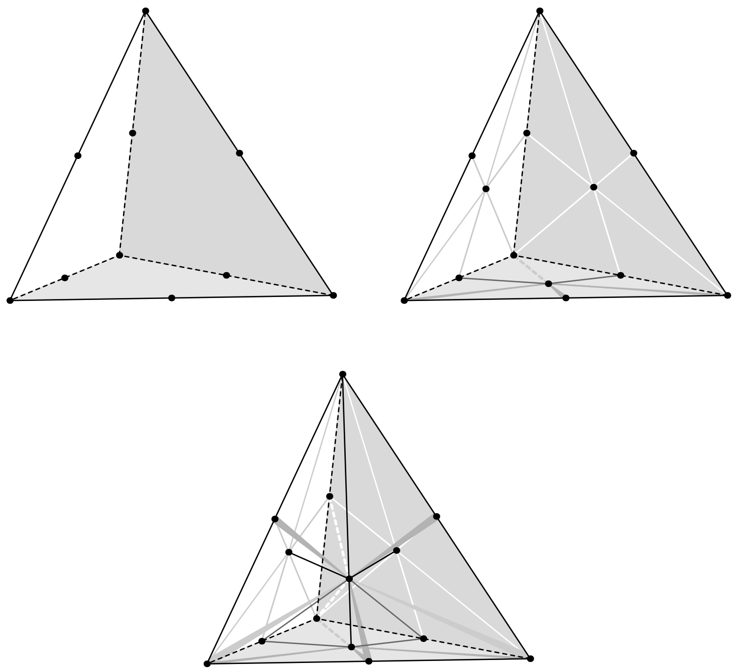

FIGURE 10. The three last steps of the decomposition in dimension 3

Remark 15. *The intersection of two barycentric simplexes is a lower dimensional simplex: it is the closure of a common face containing the barycentre of the polytope.*

S_i is the simplex whose vertexes are the point $v_{i,0},\dots,v_{i,n}$, where $v_{i,n}=p_n$ is the barycentre of $\mathcal{P}$, and for $k=n-1,\dots,0$, $v_{i,k}$ is the barycentre of a k-dimensional face, always on the boundary of the face $v_{i,k+1}$ belongs to (see Figure 11).

To $i=1,\dots,N$ we will also associate the positive cone C_i based on p_n and defined by the vectors $\varpi_{i,k}=v_{i,k}-v_{i,n}$ for $k=n-1,\dots,0$. We will call them the *barycentric cones* associated to the polytope (see Figure 12)

The convex hull in $\mathbb{R}^{n+1}$ of the $n+1$ points $(1,0,\dots,0)$, $(0,1,\dots,0)$, $\dots$, $(0,0,\dots,1)$ will be denoted by $\mathfrak{S}_n$ and called *standard n-simplex*.

We will call *standard barycentric n-simplex* of the standard n-simplex, and denote it by $\mathcal{S}_n$, the convex hull of the points (see Figure 13):

$$\widehat{v}_k := \Big(\underbrace{\frac{1}{k+1},\cdots,\frac{1}{k+1}}_{k+1 \text{ times}},\underbrace{0,\cdots,0}_{n-k \text{ times}}\Big) \text{ for } n\geqslant k\geqslant 0. \tag{7}$$

We will denote by W_n the n-dimensional hyperplane in $\mathbb{R}^{n+1}$ defined by the equation

$$x_1+\cdots+x_{n+1}=0$$

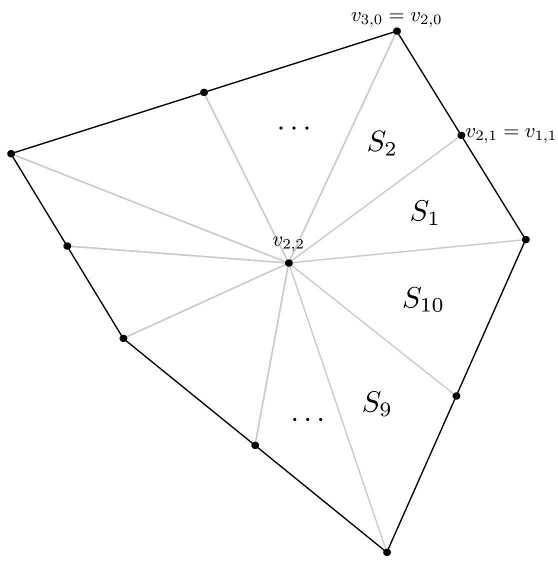

FIGURE 11. Barycentric simplexes of a polygon

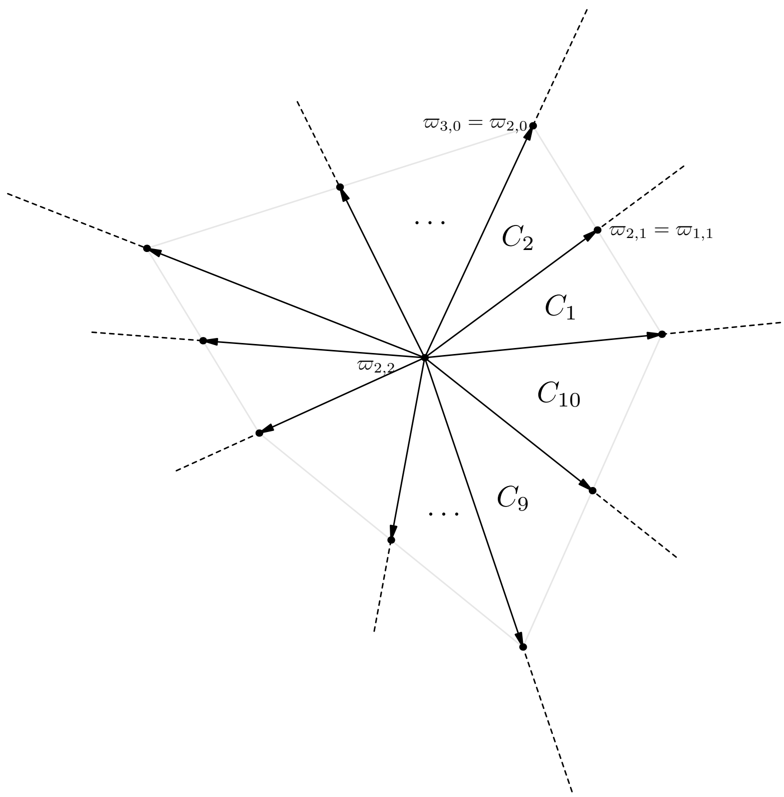

FIGURE 12. Barycentric cones of a polygon

3.2. **Embedding into the standard simplex.** We keep the notations of the previous subsection. Let L_i be the linear map sending

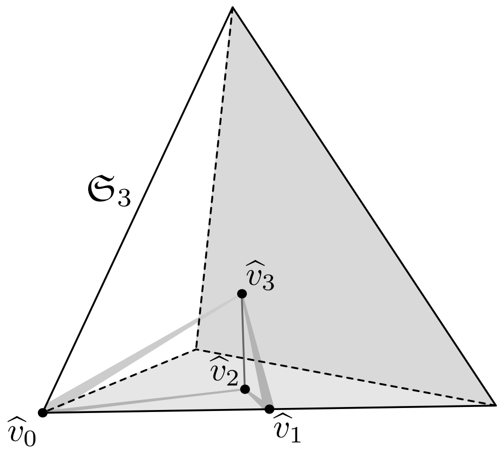

FIGURE 13. The standard barycentric 3-simplex of the 3-simplex

the barycentric simplex S_i onto the standard barycentric n-simplex $\mathcal{S}_n \subset \mathfrak{S}_n$ by mapping the point $v_{i,k}$ to $\hat{v}_k$.

Let $P_i = L_i(\mathcal{P})$ the image of the convex polytope by this linear map. L_i is an isometry between the Hilbert geometries of P_i and $\mathcal{P}$, in other words for any x in the interior of $\mathcal{P}$ we have (identifying L_i with its differential)

$$F_{P_i}\big(L_i(x), L_i(v)\big) = F_{\mathcal{P}}(x, v).$$

This way, $\mathcal{S}_n$ is a common flag conical neighborhood of both P_i and $\mathfrak{S}_n$ and by theorem 8 we obtain:

Property 16. *There exists a constant k_i such that for any point x of the standard barycentric simplex $\mathcal{S}_n$ and any vector v one has*

$$\frac{1}{k_i} \cdot F_{P_i}(x, v) \leqslant F_{\mathfrak{S}_n}(x, v) \leqslant k_i \cdot F_{P_i}(x, v)$$

3.3. **From the standard simplex to W_n.** Let $\Phi_n \colon \mathfrak{S}_n \to W_n \simeq \mathbb{R}^n \subset \mathbb{R}^{n+1}$ defined by

$$\Phi_n(x_1, \cdots, x_{n+1}) = (X_1, \cdots, X_{n+1}) = \left(\ln\Big(\frac{x_1}{g}\Big), \cdots, \ln\Big(\frac{x_{n+1}}{g}\Big)\right)$$
$$\text{with } g = (x_1 \cdots x_{n+1})^{1/n+1}$$

Thanks to P. de la Harpe [dlH93], we know that Φ_n is an isometry from the simplex $\mathfrak{S}_n$ into W_n endowed with a norm whose unit ball is a centrally symmetric convex polytope.

For our purpose, let us remark that the image of the standard barycentric simplex $\mathcal{S}_n$ by Φ_n is the positive cone of W_n of summit at the origin and defined by the vectors

$$(8) \quad \widetilde{v}_k := \Big(\underbrace{n-k, \cdots, n-k}_{k+1 \text{ times}}, \underbrace{-(k+1), \cdots, -(k+1)}_{n-k \text{ times}}\Big) \text{ for } n > k \geqslant 0$$

and we denote it by $\widetilde{\mathcal{C}_n}$ and call it *standard n-cone.*

Now for any polytopal convex set $\mathcal{P} \in \mathbb{R}^n$, consider the map M_i which maps the standard n-cone $\widetilde{\mathcal{C}_n}$ into the barycentric cone C_i based on p_n, by sending the origin to p_n and the vector $\widetilde{v_k}$ to the vector $\varpi_{i,k}$.

3.4. **Conclusion.** We can now define our bi-lipschitz map

$$F\colon (\mathcal{P}, d_{\mathcal{P}}) \to (\mathbb{R}^n, ||\cdot||)$$

in the following way.

$$\forall x \in S_i, \quad F(x) = M_i\Big(\Phi_n\big(L_i(x)\big)\Big) \tag{9}$$

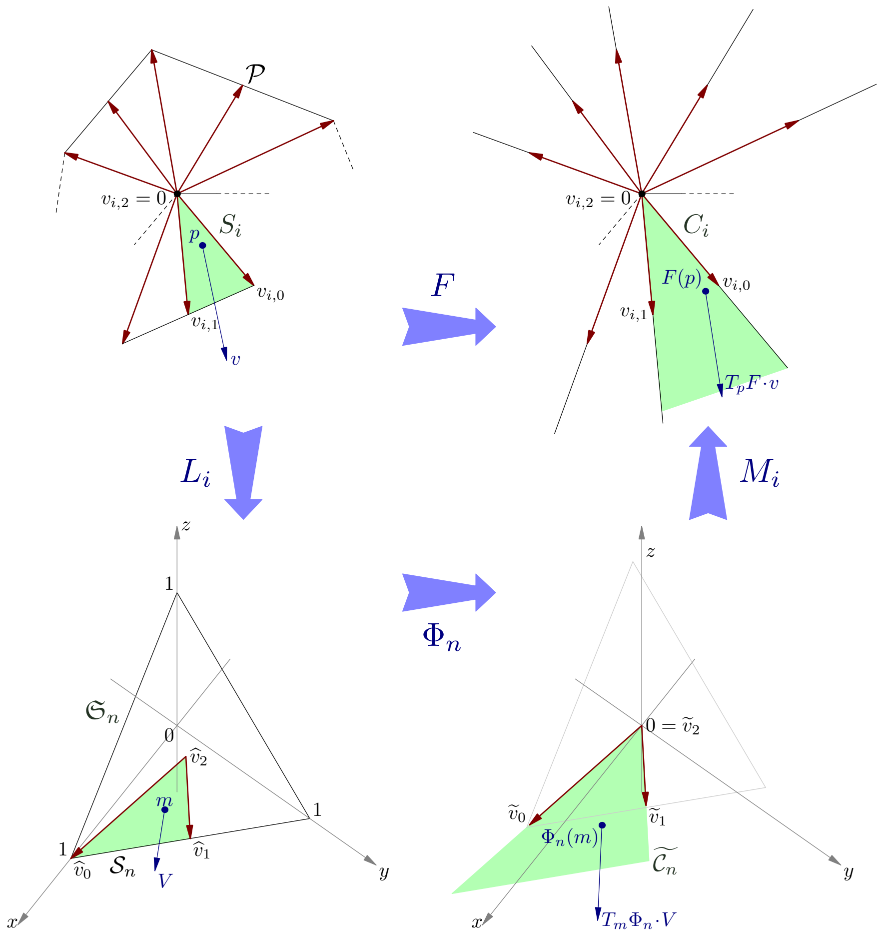

FIGURE 14. The application F in dimension 2 illustrated

Thanks to the remark 15, if $x \in \mathcal{P}$ is a common point of S_i and S_j, then necessarily $L_i(x) = L_j(x)$ thus,

$$\Phi_n\big(L_i(x)\big) = \Phi_n\big(L_j(x)\big) = y$$

and y is on boundary of the cone $\widetilde{\mathcal{C}_n}$. Now $M_i(y) = M_j(y)$, because M_i and M_j send the corresponding boundary cone of $\widetilde{\mathcal{C}_n}$ to the respective common boundary cone of the cell-cones C_i and C_j in the same way. In other words,

$$\forall x \in S_i \cap S_j, \quad L_i(x) = L_j(x)$$

and

$$\forall z \in C_i \cap C_j, \quad M_i^{-1}(z) = M_j^{-1}(z)$$

thus F is well defined and it is a bijection.

To prove that it's bi-lipschitz, we use the fact that line segments are geodesic and that both spaces are metric spaces.

Hence let p and q be two points in the polytope $\mathcal{P}$. Then there are $M \in \mathbb{N}$ points $(p_j)_{j=1,\ldots,M}$ on the segment $[p, q]$ such that $p = p_1$, $q = p_M$, and each segments $[p_j, p_{j+1}]$, for $j = 1, \ldots, M-1$, belongs to a single cell-simplex S_j of the cell-simplex decomposition of $\mathcal{P}$.

Thanks to the key-lemma 16, and the fact that all norms in $\mathbb{R}^n$ are equivalent, we know that for each j, there is a constant k'_j such that, for $x, y \in S_j$, on has

$$\big\|F(x) - F(y)\big\| \leqslant k'_j \cdot d_{\mathcal{P}}(x, y)$$

Applying this to p_j, p_{j+1} for $j = 1, \ldots, M-1$, we obtain

$$\sum_{j=1}^{M-1} \big\|F(p_j) - F(p_{j+1}\big\| \leqslant (\sup_i k'_i) \cdot d_{\mathcal{P}}(p, q)$$

where the supremum is taken over all cells of the decomposition, then from the triangle inequality one concludes that

$$\big\|F(p) - F(q)\big\| \leqslant (\sup_i k'_i) \cdot d_{\mathcal{P}}(p, q).$$

Starting from a line from $F(p)$ to $F(q)$ and taking it inverse image after decomposing it in segments, which are all in a single barycentric-cone, we obtain in the same way the inverse inequality

$$d_{\mathcal{P}}(p, q) \leqslant (\sup_i k'_i) \cdot \big\|F(p) - F(q)\big\|.$$

4. Hilbert geometries bi-lipschitz to a normed vector space

Some remarks and references on the reciprocal of theorem 1 are given.

Colbois and Verovic in [CV] prove that a Hilbert geometry which quasi-isometrically embeds into a normed vector space is the Hilbert geometry of a Polytope. Notice that in their paper they state a weaker result but actually prove this strongest statement.

In our paper [Ver] we prove that the asymptotic volume of a Hilbert geometry is finite if and only if it is the geometry of a polytope. Therefore, this allows us to conclude, without the strongest result of Colbois and Verovic, that a Hilbert geometry bi-lipschitz to a normed vector space comes from a polytope.

References

[Ben03] Y. Benoist. Convexes hyperboliques et fonctions quasi symétriques. *Publ. Math. Inst. Hautes Études Sci.*, 97:181–237, 2003

[Ben06] Y. Benoist. Convexes hyperboliques et quasiisométries. (Hyperbolic convexes and quasiisometries.). *Geom. Dedicata*, 122:109–134, 2006.

[BBV10] G. Berck, A. Bernig and C. Vernicos, *Volume entropy of Hilbert Geometries*, Pacific. J. of Math., **245**(2):201–225, 2010.

[AB09] A. Bernig, *Hilbert Geometry of Polytopes*, Archiv der Mathematik **92** (2009), 314-324.

[CV06] B. Colbois and C. Vernicos, *Bas du spectre et delta-hyperbolicité en géométrie de hilbert*, Bulletin de la Société Mathématique de France **134** (2006), 357–381.

[CV07] ———, *Les géométries de hilbert sont à géométrie locale bornée*, Annales de l'Institut Fourier **57** (2007), no. 4, 1359–1375.

[CVV08] B. Colbois, C. Vernicos, and P. Verovic, *Area of Ideal Triangles and Gromov Hyperbolicity in Hilbert Geometries*, Illinois Journal of Mathematics, **52**(1):319-343 (2008).

[CVVa] ———, *Hilbert geometry for convex polygonal domains*, preprint 2008, to appear in Journal of Geometry. arXiv:0804.1620v1 [math.DG].

[CV04] B. Colbois and P. Verovic, *Hilbert geometry for strictly convex domains*, Geom. Dedicata **105** (2004), 29–42. MR 2 057 242

[CV] ———, *Hilbert domains quasi-isometric to normed vector spaces* preprint 2008, arXiv:0804.1619v1 [math.MG].

[FK05] T. Foertsch and A. Karlsson, *Hilbert Geometries and Minkowski norms*, Journal of Geometry, Vol.83, No. 1-2, 22-31 (2005)

[dlH93] P. de la Harpe, *On Hilbert's metric for simplices*, Geometric group theory, Vol. 1 (Sussex, 1991), Cambridge Univ. Press, Cambridge, 1993, pp. 97–119.

[KN02] A. Karlsson and G. A. Noskov, *The Hilbert metric and Gromov hyperbolicity*, Enseign. Math. (2) **48** (2002), no. 1-2, 73–89. MR 2003f:53061

[Kay67] D. C. Kay. The ptolemaic inequality in Hilbert geometries. *Pacific J. Math.*, 21:293–301, 1967.

[Lin07] B. C. Lins *Asymptotic behavior and Denjoy-Wolff theorems for Hilbert metric nonexpansive maps*, PhD dissertation, Rutgers University, 2007.

[SM02] E. Socié-Méthou, *Caractérisation des ellipsoïdes par leurs groupes d'automorphismes*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 4, 537–548. MR 1 981 171

[SM04] ———, *Behaviour of distance functions in Hilbert-Finsler geometry*, Differential Geom. Appl. **20** (2004), no. 1, 1–10. MR 2004i:53112

[Ver09] C. Vernicos, *Spectral Radius and Amenability in Hilbert Geometries* , Houston journal of Maths. 3**5**(4):1143-1169 (2009).

[Ver] ———, *Asymptotic volumes of Hilbert geometries*, preprint 2011.

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